

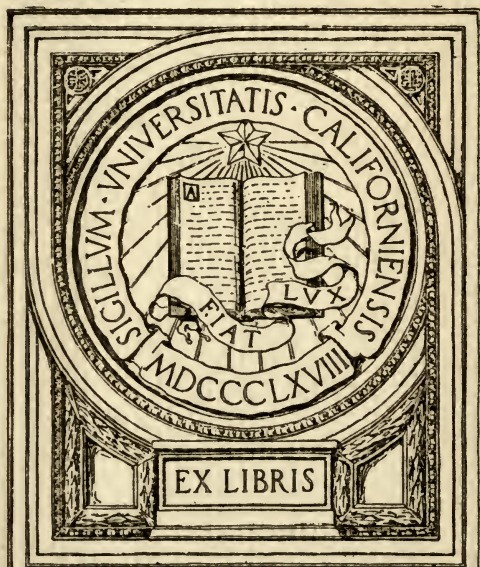
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MATHEMATICAL TRACTS.

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# MATHEMATICAL TRACTS.



## PART I.

BY

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## TRACT I.

### ON THE BASES OF GEOMETRY WITH THE GEOMETRICAL TREATMENT OF $\sqrt{-1}$ .

#### CONTENTS.

- (1) On the Treatment of Ratio between Quantities Incommensurable.
- (2) Primary Ideas of the Sphere and Circle. Poles of a Sphere.
- (3) Definition and Properties of the Straight Line.
- (4) Definition and Properties of the Plane.
- (5) Parallel Straight Lines based on the Infinite Area of a Plane Angle.
- (6) On the Volume of the Pyramid and Cone.

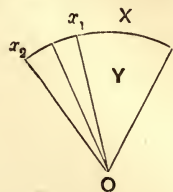
#### I. THE RATIO OF INCOMMENSURABLES.

1. IN arithmetic the first ideas of ratio and proportion, and the laws of passage from one set of 4 proportionals to another, ought to be learned, as preliminary to geometry; but in geometry the doctrine of incommensurables requires a special treatment, unless the learner be well grounded in the argument of infinite converging series. Repeating decimals may perhaps suffice. Another, possibly better way, is open by the introduction of VARIABLE quantities, which will here be proposed.

2. Nothing is simpler than to imagine some geometrical quantity to vary in shape or size according to some prescribed law. This must imply at least *two* quantities varying *together*. Thus, if an equilateral triangle change the length of its *side*, its *area* also changes. If the radius of a circle increase or diminish, so does the length of the circumference. In general two magnitudes *X* and *Y* may *vary together*: they may be either the same in kind,—as the radius and circumference of a circle is each *a length*; or the two may be different in kind, say, a length and an area. In general it is a

convenient notation to suppose that when  $X$  changes to  $X'$ ,  $Y$  changes to  $Y'$ .

3. Again, if  $X$  receive successive additions  $x_1, x_2, x_3, \dots, x_n$ , the corresponding additions (if additions they be) to  $Y$  are well denoted by  $y_1, y_2, y_3, \dots, y_n$ . An obvious and simple case, if it occur, will deserve notice; namely, if the two variables are so regulated, that equality in the first set of additions (i.e.  $x_1 = x_2 = x_3 = \dots = x_n$ ) induces equality in the second set; (i.e.  $y_1 = y_2 = y_3 = \dots = y_n$ ). The variables  $X$  and  $Y$  are then said to *increase uniformly*. As an obvious illustration, suppose  $X$  to be the *arc* of a circle, and  $Y$  the area of the *sector* which it bounds, evidently then if the arcs  $x_1, x_2$  are equal increments of the arc  $X$ , the sectors  $y_1, y_2$  which are bounded by  $x_1, x_2$  will be equal increments of  $Y$ . Then the arc  $X$  and the sector  $Y$  increase together *uniformly*.



4. We may now establish a theorem highly convenient for application in geometry, alike whether quantities are commensurable or incommensurable.

**THEOREM.** "If  $X$  and  $Y$  are any two connected variables, which begin *from zero* together, and increase *uniformly*; then  $X$  varies proportionably to  $Y$ . In other words, if  $Y$  become  $Y'$  when  $X$  becomes  $X'$ , then  $X$  is to  $X'$  as  $Y$  is to  $Y'$ ."

*Proof.* First, suppose  $X$  and  $X'$  commensurable, and  $\xi$  a common measure, or  $X = m \cdot \xi$  ( $m$  times  $\xi$ ) and  $X' = n\xi$ . We may then suppose  $X$  and  $X'$  made up by repeated additions of  $\xi$ . Every time that  $X$  has the increment  $\xi$ ,  $Y$  will receive a uniform increment which we may call  $\nu$ ; then  $Y$  is always the same multiple of  $\nu$  that  $X$  is of  $\xi$ ; thus the equation  $X = m\xi$  implies  $Y = m\nu$ , and  $X' = n\xi$  implies  $Y' = n\nu$ . Hence  $X : X' = m : n = Y : Y'$ .

Next, when  $X'$  is *not* commensurate with  $X$ , yet  $\xi$  is some *submultiple* of  $X$ , such that  $n\xi = X$ , and  $X'$  contains  $\xi$  more than  $m$  times, but less than  $(m + 1)$  times; evidently we *cannot* have

$$X : X' = Y : Y'$$

(when the four magnitudes are presented to us) unless, as a first condition, on assuming  $n\nu = Y$ , we find  $Y'$  to contain  $\nu$  more than  $m$  times and less than  $(m + 1)$  times: and unless this condition were fulfilled,  $X$  and  $Y$  would *not* increase *uniformly*. We may therefore

assume  $X_2X_3$  on opposite sides of  $X'$ , with values

$$X_2 = m\xi, X_3 = (m+1)\xi;$$

likewise  $Y_2Y_3$  on opposite sides of  $Y'$ , with values

$$Y_2 = mv, Y_3 = (m+1)v.$$

Then by the first case we have  $X : X_2 = Y : Y_2$  and  $X : X_3 = Y : Y_3$ . But  $X_3 - X_2 = \xi$ , and  $Y_3 - Y_2 = v$ . Let  $n$  perpetually increase, then  $\xi$  and  $v$  perpetually lessen.  $X_2$  and  $X_3$  run together in  $X'$ ,  $Y_2$  and  $Y_3$  run together in  $Y'$ . Thus each of the ratios  $X : X_2$  and  $X : X_3$  falls into  $X : X'$ , and each of the ratios  $Y : Y_2$ ,  $Y : Y_3$  falls into  $Y : Y'$ . Inevitably then,  $X : X' = Y : Y'$ , even when these last are incommensurate. Q.E.D.

## II. PRIMARY IDEAS OF THE SPHERE AND CIRCLE.

For the convenience of *beginners*, POSTULATES may be advanced concerning the straight line and the plane, as well as concerning parallel straight lines. But in the second stage of study the whole topic ought to be *treated anew* from the *beginning*: a task which is here assumed.

### *On Length and Distance.*

**THEOREM.** "All lengths are numerically comparable." To make this clear, it is simplest to imagine a thread indefinitely thin, flexible and inextensible. This, if applied upon any given line, will become an exact measure of its length; and if any two lines be then measured by two threads, the threads are directly comparable, shewing either that they are equal, or that one is longer than the other and how much longer. Hereby we safely assert the same fact concerning any two given lengths.

Obviously, length is *continuous* magnitude: which means, that if a point  $P$  run along from  $A$  to  $B$ , the length  $AP$  passes *through all* magnitude from zero to  $AB$ .

**THEOREM.** Any two given points in space may be joined *either* by one path which is shorter than any other possible, *or* by several equal paths than which none other is so short. For of all possible paths joining them some must be needlessly long; yet unless there is some limit to the shortening, the distance would be *nil*; the points would not be two, but would coincide and become one.



DEF. A shortest path that joins two points in space gives a measure of their DISTANCE. The same argument applies, if the two given points and the line that joins them must lie on a given surface; or again, if two surfaces that do not touch be given, and we speak of the shortest distance of the two surfaces.

Assume a fixed point  $A$  and a second point  $S$  so movable as always to be *at the same distance* from it. It will be able to play all round  $A$ : therefore *its locus* will be a surface enclosing  $A$ . The solid mass enclosed is called a SPHERE (*Globe* or *Ball*) and  $A$  its CENTRE.

THEOREM. "Every point *outside* the sphere is *further* from the centre and every point *within* the sphere is *nearer* to the centre, than are the points on the surface." For if  $T$  be an exterior point, every path joining  $T$  to  $A$  must pierce the surface in some point  $S$ ; therefore the path  $TSA$  is longer than  $SA$  by the interval  $TS$ . Again, if  $R$  be within the sphere, we may imagine an interior sphere whose surface is at the common distance  $AR$  from  $A$ . Then  $S$  being exterior to the new sphere,  $SA$  is longer than  $RA$ ; that is,  $R$  within the sphere of  $S$  is nearer to  $A$  than is the locus of  $S$ . Q. E. D.

DEF. Two such concentric spheres enclose within their surfaces a solid called a spherical *shell*.

THEOREM. "The two surfaces are equidistant, each from the other." For if the shortest distance from a point  $S$  to the inner surface is the path  $SR$ , *symmetry* all round shews at once that if from a second point  $S'$  the shortest path will be  $S'R$ , the two distances  $SR$ ,  $S'R$  will be equal. Indeed it is not amiss to remark, that if any spherical surface be rigidly attached to its centre, the entire surface may glide on its own ground without disturbing its centre, because the distances  $SA$ ,  $S'A$  nowhere change. Hence also we may justly imagine the spherical *shell* to glide *on its own ground*, while the centre suffers no displacement, and any shortest path  $S'R$  joining the opposite sides of the shell may assume the place which was previously held by  $SR$ . *Actual superposition* thus attests equality of distance.

THEOREM. "If a spherical surface be given, its centre is determined." For if an inner point  $R$  be assumed at a given distance  $D$  from the surface, its *locus* is an *interior* continuous surface. Within this, at distance  $D'$ , imagine a point  $R'$  to generate a second con-

tinuous surface, and it will be interior to the preceding and so continually. The series of surfaces must then necessarily converge towards a single point, which will be the centre of the given surface, because the sum of the distances is the same, from whichever point we calculate. The same argument proves that all the surfaces are concentric *spheres*.

### *Poles of a Sphere.*

THEOREM. "To every point on a sphere one opposite point lies at the longest distance along the surface." For if the point  $P$  be given, and we take a point  $S$  at any distance from  $P$  *along the surface*, and suppose  $S$  to vary under the sole condition that its distance from  $P$  (along the surface) shall not change, the locus of  $S$  is a self-rejoining line enclosing  $P$ . (We call this a circle.) Next, *beyond*  $S$ , along the surface, take a new point  $T$ , which moves without changing its distance from  $S$  and from  $P$ . This generates an outer circle, cutting off a part of the surface which was beyond the circle of  $S$ . Beyond this we may similarly form a third circle, and this series of circles ever lessening the *finite* area beyond it, will necessarily converge towards a point  $Q$  on the sphere.  $P$  will then be farther from  $Q$  (along the sphere) than any of these parallel circles. We call  $P$  and  $Q$  *opposite poles* of the sphere. The distance between them is evidently the *half girth* of the sphere.

Every point on the sphere has not only its own opposite pole; but also its system of equidistant (or parallel) circles. The middle one of these (that is, the one equidistant from the two poles), is called their equator.

If in an equator whose poles are  $P$  and  $Q$ , you fix any point  $C$ , and then proceeding half round the equator fix a second point  $D$ ,  $C$  and  $D$  are evidently opposite poles.

If you imagine a sphere to glide on its own ground, with centre unmoved, you may suppose  $P$  to pass over to the site held previously by  $Q$ . This carries  $Q$  to the place previously held by  $P$ . Thus the poles are exchangeable, while the sphere *as a whole* is unchanged and the same equator is attained.

THEOREM. "If  $P$  and  $R$  be any two points on a sphere that are *not* opposite poles, one equator, and one only, passes through them both."

*Proof.* Through  $P$  and its opposite pole  $Q$  (just as above through the poles  $C$  and  $D$ ) an equator may pass. If this half equator  $PQ$  become rigid and be rigidly attached to the fixed centre  $A$ , it still may sweep over the spherical surface (without change of  $P$  or  $Q$ ) until it passes through  $R$ ; but after passing once through  $R$ , it does not come back to it, except in a second revolution. Q. E. D.

### III. POINTS LYING EVENLY.

In Simson's Euclid, the line whose *points lie evenly* is called STRAIGHT; but the phrase "lying evenly" is not explained. We can now explain it.

When the two poles  $P$  and  $Q$ , and the centre  $A$ , all remain unchanged, nevertheless each of the parallel circles associated with  $P$  and  $Q$  can glide on their own ground. Evidently then, if  $P$  and  $A$  be fixed, this suffices to fix  $Q$ . In fact while each circle spins round its own line,  $Q$  can only spin *round itself*. Also, to fix  $P$  and  $Q$  fixes  $A$ .—These parallel circles excellently define to us the idea of rotation, which is a constrained motion, still possible, even when  $P$ ,  $A$ ,  $Q$  are all fixed. Now suppose that a line  $PMQ$  internal to the sphere rigidly connects  $P$  with  $Q$ . Then if the system revolve round  $P$ ,  $A$ ,  $Q$ ,  $PMQ$  may generate a self-rejoining surface within the sphere. Again within this new surface a rigid line  $PNQ$  may connect  $P$  with  $Q$ , and the line  $PNQ$  by rotation round  $P$ ,  $A$ ,  $Q$  may generate a third surface interior to the preceding; and so on continually. Since there is no limit to the constant thinning of the innermost solid, we see that a mere line without thickness connects  $P$  with  $Q$  and passes through  $A$ , which line is *interior to all the solids* and during rotation remains immovable. It is called an axis, and can only turn about itself. Hence every point in this axis *lies evenly* between  $P$  and  $Q$ .

And since  $P$  and  $Q$  may represent any two points in space, we now discover that between any two there is a unique line lying evenly. This continuous line, while we talk of rotation round it, is entitled an *axis*; but ordinarily we call it simply STRAIGHT.

#### *On the Straight Line and its "Direction."*

We now infer that

1. Any two points in space can be joined by a straight line.
2. Every part of a straight line is straight.



3. A unique straight line is *determined*, when its two end-points are given.

4. Any part of a straight line, if removed, may take the place of any equal part of the same. Hence it easily follows that a straight line, *gliding along itself*, will prolong itself indefinitely far, either way, along a determinate course.

We are now able to sharpen our idea of direction. Hitherto we might say vaguely, "Imagine a path to proceed *in any direction*," that is, without particular guidance. But now we see, that if ever so short a straight line be drawn, it points to a definite prolongation beyond itself, of indefinite extent. This we entitle its *direction*. If this direction be changed, a deviation there occurs, and a sharp corner is recognized at the point of deviation. The *amount* of deviation suggests a new kind of magnitude, which will presently need attention. Now it suffices to remark on the case in which a new line  $AZ$  deviates equally from a previous line  $PA$  and from  $AQ$  the prolongation of  $PA$ . The equality is tested by imagining  $AZ$  to become an axis of a sphere. Then if  $P$  and  $Q$  revolve in the *same* circle,  $ZA$  is equally inclined to  $AP$  and to  $AQ$ . It is called perpendicular to  $PAQ$ . Evidently  $Z$  (on the sphere) is at the distance of a quarter girth from every point of the equator traced by  $P$  and  $Q$ .

#### IV. THE PLANE.

We return to the sphere. When any two poles  $P, Q$  are joined by a straight line, it has been seen that this passes through the centre  $A$ . The line  $PAQ$  is called a *diameter* of the sphere, and its half ( $AP$  or  $AQ$ ) is called a *radius*.

Evidently all the radii of the same sphere are equal; and of different spheres the greater the radius, the greater the sphere.

If an equator  $CDEC$  is midway between the poles  $P, Q$ , and  $D$  is the pole opposite to  $C$ , then as the diameter  $PQ$ , so too the diameter  $CD$ , passes through centre  $A$ . This is true, whatever point in the equator is assumed for  $C$ . Therefore  $CAD$  is a varying diameter, whose extremities trace out the equator, while the diameter traces out a surface in which the equator lies. This surface is called a **PLANE**, and in particular is the plane of the equatorial circle.

It was seen that  $P$  and  $Q$  might exchange places, while the centre  $A$ , and the sphere's surface *as a whole*, remain unchanged. Necessarily also the plane of the equator remains unchanged. It is

then symmetrical on its opposite sides, or in popular language, the plane turns *the same face* towards  $P$  as towards  $Q$ .

The axis  $PAQ$  is called perpendicular to *the plane* of the equator, being perpendicular to *every radius* of the equatorial circle.

**THEOREM.** "No other line but  $AP$  can be perpendicular to the plane of the equator."

*Proof.* For if  $AR$  be some other radius of the sphere, some one of the parallel circles, whose pole is  $P$ , passes through  $R$ , and every point of this circle is nearer to the equatorial circle than is the pole  $P$ . Therefore the distance of  $R$  from the equator is less than a quarter of the sphere's girth, a fact which shews  $RA$  *not* to be perpendicular.

**THEOREM.** "Through any two radii  $AP$ ,  $AR$  of a sphere, that are not in the same straight line, one plane and one only may pass."

It has been seen that through  $P$  and  $R$  only one equator can pass. The plane of this equator is the plane that passes through the two radii.

### *Cardinal Property of the Plane.*

**THEOREM.** "If  $M$  and  $N$  are any two points in a plane, no point in the straight line which joins  $M$  and  $N$  can *lie off* the plane on either side."

Symmetry suffices to establish this truth. Our hypothesis supplies *data* to fix what line is meant by  $MN$ , but gives no reason why any point of it should lie off the plane on one side *rather than* on the other; for the whole line is determined by merely the extreme points  $M$ ,  $N$ , of which neither can guide any point towards  $P$  rather than towards  $Q$ . Thus there is *no adequate reason* for deviation towards either side.

Symmetry of data is in other mathematical topics accepted as *an adequate argument* for symmetry of results. Otherwise, "the want of *sufficient reason* for diversity" passes as refutation of alleged diversity. Therefore the argument here presented has nothing really novel.

We have now a new method of generating a plane that shall pass through two intersecting straight lines  $LM$ ,  $MN$ . Along  $ML$  let a point  $E$  run, and along  $MN$  similarly a point  $F$ . Join  $EF$  while the motion of  $E$  and of  $F$  continues. Then  $EF$  (by the last



Theorem) always continues to rest on the plane  $LMN$ . This mode of generating the plane supersedes the idea of rotation. For simplicity we might suppose  $\overline{ME} : \overline{MF}$  to retain a fixed ratio.

THEOREM. "A plane has no unique point or centre."

For if we start from given spherical radii  $AP$ ,  $AR$  through which passes an (equatorial) plane, in  $AP$  take  $M$  arbitrarily, and in  $AR$  take  $N$  arbitrarily. Then we have seen that the *locus* of the moving line  $MN$  is our given plane. But again, in this plane take a fixed point  $O$ , and join  $O$  to *fixed* points  $M$  and  $N$ . Then from the lines  $OM$ ,  $ON$  we can (as in the last) generate the very same plane, which can glide on its own ground as the sphere did; thus the point  $A$  can pass to  $O$  without changing the ground or surface as a whole. The plane is infinite, the sphere is finite; but as with the sphere, so with the plane, no point of the surface is unique.

After this, no impediment from logic forbids our passing to the received routine of Plane Geometry, until we are arrested by the difficulty of parallel straight lines, to which I proceed, after one remark on the definition of *an angle*.

Above, a sharp corner or turn was identified with deviation, or change of direction. In geometry it has the name of an angle, and we measure its magnitude by aid of the circular arc which it subtends at the centre or by the sector of that arc. But no insuperable logic forbids our estimating the magnitude of an angle by the portion of *the infinite area* which it intercepts from a plane; which indeed is suggested by a perpetual elongation of the radius of the circle whose *sector* was assumed as measure of the angle.

Monsieur Vincent in Paris (1837) adopted this definition as adequate to demonstrate the equivalent of Euclid's Twelfth Axiom without any new axiom at all. Has this method received due attention in England?

Monsieur Vincent was not the first to suggest accepting *the infinite plane* area cut off by two intersecting straight lines, as the measure of the *angle* which they enclose: but perhaps he was the first to introduce the method into a treatise on Elementary Geometry, that obtained acceptance in so high an institution as the University of France.

*Two lemmas* alone are wanted, and these every beginner will find natural.

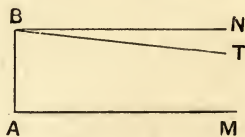
LEMMA I. "Every angle is a finite fraction of a right angle;" that is, some finite multiple of it exceeds  $90^\circ$ . For the circular arc which subtends it, is always some finite fraction of the quarter of the circumference.

DEF. When two straight lines  $AM$ ,  $BN$  in the same plane are both perpendicular to a third straight line  $AB$ , we call that portion of the plane area which is enclosed between  $MA$ ,  $AB$  and  $BN$  a BAND.

LEMMA II. Then, I say, whatever the breadth ( $AB$ ) of the band, the area of the band is *less than* any finite fraction of a right angle.

*Proof.* Prolong  $AB$  indefinitely to  $X$ , and along it take any number of equal lengths  $AB = BC = CD = DE$ , &c., and through  $C$ ,  $D$ ,  $E$ ... draw perpendicular to  $ABCDE$ ... straight lines  $CO$ ,  $DP$ ,  $EQ$ , &c. Evidently then the successive bands are equal, by superposition. Thus, whatever multiple of the first band be *deducted* from the plane area marked off by the right angle  $MAX$ , the loss is insensible; for, as remainder, we find the area marked off still by a right angle (such as  $QEX$ , if only four bands were deducted). Any two right angles embrace areas which can be identified by superposition, and have no appreciable difference. The matter may be concisely summed up by remarking, that every band is infinite in *one* direction only,—say, horizontally—but the area embraced by any right angle is infinite in *both* directions, horizontally and also vertically. Thus it is no paradox to say, that no finite multiple of the band can, by its deduction from the area of the right angle, lessen that infinite area in our estimate. Q.E.D.

Euclid's Twelfth Axiom is now an immediate corollary; viz. If  $MABN$  be any band; and, *within* the right angle  $NBA$ , any straight line  $BT$  be drawn, it can be prolonged so far as to meet the prolongation of  $AM$ . For the angle  $NBT$  is a finite fraction of a right angle, while the band  $MBAN$  is less than any finite fraction of the same; hence the angle  $NBT$  is greater than the band  $MABN$ , but unless  $BT$  crossed  $AM$  this would be false. Thus of necessity the two lines do cross, as we asserted.



I cannot see *any new* axiom involved in this proof: therefore I am forced to abandon several other specious methods and give it

preference. Surely we may bow to the authority of the University of France in such a matter.

### *On the Volume of Pyramids and Cones.*

The treatment of this topic in Euclid is very clumsy. It demands and it admits much improvement.

1. For parallelepipeda prove first, that if *two* such solids differ solely in the length of one edge, which we may call  $x$  in the one and  $a$  in the other, then their volumes are in the proportion of  $x : a$ .

2. Next, if they have a solid angle in common, but the edges round it are in one  $x, y, z$ , and in the other  $a, b, c$ , then the two volumes are in the proportion of  $xyz : abc$ .

3. After this it is easily shewn that parallelepipeda on *the same base* and *equal height* have equal volumes.

4. Therefore finally, that the volume of a parallelepipedon is measured by its base  $\times$  its height. COR. The same is true of any prism.

From this we proceed to approximate to the volume of a pyramid.

5. Divide the height ( $h$ ) into ( $n$ ) equal parts by  $(n-1)$  planes all parallel to the base ( $B$ ). Establish, on these  $(n-1)$  bases, upright walls, and you will find you have constituted a double system of prisms, one interior to the pyramid, one exterior; the latter has the lowest prism in excess of the other system. Every base is similar to every other, by the nature of a pyramid. The volume here of every prism is  $\frac{h}{n} \times$  its base, the number  $n$  and  $\frac{h}{n}$  being the same for all, but the base varying.

6. The base whose distance from the vertex is  $\frac{r}{n} \cdot h$ , is to the original ( $B$ ) as  $r^2 : n^2$ ; hence its area is  $\frac{r^2}{n^2} \cdot B$ , which gives for the volume of the prism standing on it  $\left(\frac{r^2}{n^2} \cdot B\right) \cdot \frac{h}{n}$ . Hence the sum of the volumes of the *external* prisms is  $\frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^2} \cdot \frac{h}{n} \cdot B$ , and by omitting  $n^2$  from the numerator of the larger fraction we obtain



the sum of volumes for the *internal* prisms. Now since  $\frac{n^2 \cdot h \cdot B}{n^3}$  vanishes when  $h, B$  are finite and  $n$  infinite, the difference of the two systems of prisms vanishes when  $n$  is infinite. But the volume of the pyramid is less than the exterior system and greater than the interior; hence each system has the volume of the pyramid for *its limit*, when  $n$  increases indefinitely.

7. Let  $\mu$  be the *unknown* numerical limit to which the fraction  $\frac{1^2 + 2^2 + 3^2 + \dots n^2}{n^3}$  approximates when  $n$  thus increases. Then the volume of the pyramid  $= \mu \cdot h \cdot B$ . Since  $\mu$  is the limit of a numerical fraction, which remains the same, *whatever the form of the pyramid's base*, we shall know the value of  $\mu$  for all pyramids, if we can find it in one. Meanwhile the result  $V = \mu \cdot h \cdot B$  at once shews that pyramids with equal base and equal height have equal volume, since  $\mu$  is the same for all.

8. When this theorem has been attained, we have only to divide a triangular prism into three pyramids, and instantly infer that the 3 are equal among themselves; therefore that each has a volume just  $\frac{1}{3}$  of the prism, i.e. equal  $\frac{1}{3}h \cdot B$ .

This, being proved of a pyramid whose base is a triangle, shews that the unknown  $\mu$  is there exactly  $\frac{1}{3}$ .

Hence universally  $\mu = \frac{1}{3}$ , and volume of *every* pyramid  $= \frac{1}{3}h \cdot B$ , or is equal to  $\frac{1}{3}$  of the prism which has the same base and height.

COR. Every *cone* also is one third of the *cylinder* which has the same base and height.

## TRACT II.

### GEOMETRICAL TREATMENT OF $\sqrt{-1}$ .

1. In pure algebra, concerned with number only, the symbols  $+$  and  $-$ , denoting addition and subtraction, in an early stage needed elucidation when the mark of *minus* was doubled. It is found natural that  $-(+a)$  and  $+(-a)$  should both mean  $-a$ , but that  $-(-a)$  should mean  $+a$ , and  $(-a) \cdot (-b)$  should be  $+(ab)$  surprises a beginner, and is illustrated by urging that to *subtract a debt* increases the debtor's property, and to *subtract cold* is to add heat. But as soon as we apply algebra to geometry, the symbols  $+$  and  $-$  are still better interpreted of *reverse direction*; also time past and time coming afford equally good illustration. Distinguishing *positive and negative direction* along a line, we find no mystery in the fact that to *reverse negative direction* is to make it positive, so that  $-(-a)$  gives  $+a$ , as reasonably as  $-(+a)$  gives  $-a$ . If we know beforehand whether a given distance is to be counted positively or negatively along a given axis, no ambiguity is incurred, and the sign  $+$  or  $-$  generally gives the needful information. For this reason some are apt to think of  $+a$  and  $-a$  as different numbers, instead of the same number differently *directed*. Out of this rises the learner's natural complaint when he meets  $\sqrt{-1}$  or  $\sqrt{-5}$ . "*There is no such number: you confess it is imaginary: a proposition involving it has no sense.*" So murmurs every scrupulous and wary beginner: and the teacher's reply, "Somehow we work out useful results by  $\sqrt{-1}$ ," sounds like saying: "Out of this nonsense useful truth is elicited."

2. The *first* reply to be made is: No one ought to *desire* any number for  $\sqrt{-1}$  except the unit itself; the  $\sqrt{-}$  which precedes, though a double symbol, has the force of a symbol *only*. The *next* reply is decisive,—the double symbol  $\sqrt{-}$  points to a *new direction* in geometry; namely, the direction *perpendicular* to  $+1$  and  $-1$ . But to explain this fully, it is better to make a new beginning.

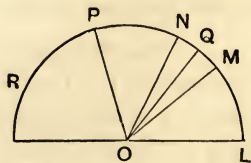
Suppose that radii issue in many directions from a fixed point in a plane, and that distances are counted along them. So long as we know along *which* radius we are to count, nothing new is involved, and of course no difficulty. Suppose one of these radii to be our ordinary *positive* axis, another to be called the  $m$  radius, and for distinction write the index  $m$  under every number to be counted along it, so that  $1_m$  is its unit, in *length* = 1 as estimated absolutely. Then we deal with  $a_m b_m c_m \dots$  along the  $m$  line, and combine  $a_m \pm b_m$ , and interpret  $5a_m, 5b_m \pm 6c_m$ , without difficulty or fear, since all are lengths to be counted along the same radius. But such a *product* as  $a_m \cdot b_n$  would need careful interpretation. In  $ab_n$  no obscurity is found, whether the  $a$  be linear or numerical. If linear, we proceed as in interpreting  $ab^3$ , though space has only three dimensions. If we put  $A$  for the value of  $ab^3$ , we attain it by the proportion  $1 : a = b^3 : A$ , so that  $A$  is the same *in kind* as  $b^3$ . Similarly if  $A' = ab_n$ , we are able to count  $A'$  along the  $n$  radius, whether  $a$  be simply numerical, or when it is linear, by aid of the ratios  $1 : a = b_n : A'$ . But if we proceed thus with  $a_m \cdot b_n$ , using the proportion

$$1_m : a_m = b_n : A',$$

we confound  $a_m b_n$  with  $ab_n$ ; for  $1_m : a_m$  is the same ratio as  $1 : a$ .

3. Mr Warren in 1826 laid a logical basis for this matter by his treatise on  $\sqrt{-1}$ , which I here substantially follow, and wonder that it is not found in all elementary works. He virtually distinguishes between proportionate *lengths* and proportionate *lines*. In the former, DIRECTION is not regarded; with the latter, it is essential. Thus if  $A, B, C, D$  are proportionate *lengths*, but are drawn along our radii, —viz.  $A$  along the positive axis,  $B$  along the  $m$  radius,  $C$  on the  $n$  radius and  $D$  on the  $p$  radius, we do not pronounce these *lines* proportional, unless also their directions justify it; that is, the  $p$  radius must be disposed towards the  $n$  radius, as is the  $m$  radius to the positive axis. This amounts to saying that the  $p$  line must lie on the same side of the  $n$  line, and at the same angular distance from it, as the  $m$  line compared with the positive axis. Then, if  $OL, OM, ON, OP$  be the 4 radii, and  $LMNP$  a circular arc, we need that the arc  $PN$  shall = arc  $ML$  before we admit that the *units*  $OL, OM, ON, OP$  are proportionate *lines*.

After this condition of the *directions* is fulfilled, we concede that



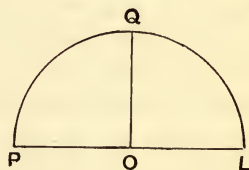


the proportionate *lengths* counted along them are also proportionate *lines*.

If the arc  $NP = \text{arc } LM$ , add  $MN$  to both, then  $\text{arc } PM = \text{arc } NL$ . This enables us to exchange the second and third terms in the proportion, agreeably to the process called *Alternando*. Also the arc  $PL = PN + NL = ML + NL$ . Hence if we count from  $L$ , and call  $\text{arc } LM = m$ ,  $\text{arc } LN = n$ ,  $\text{arc } LP = p$ , the test of necessary direction is  $p = m + n$ .

The simplest case is, when the four proportionals become three by the second and third coalescing, as if  $M$  and  $N$  run together in  $Q$ . Then if  $\text{arc } LQ = \text{arc } QP$ , we have  $OL : OQ = OQ : OP$ . If further  $\text{arc } PR = \text{arc } PQ$ , then  $OQ : OP = OP : OR$ ; and so on.

4. Apply now this to the case in which the arcs  $PQ$  and  $QL$  are both quadrants. Then  $OQ$  is the mean proportional between  $OL (= 1)$  and  $OP = (-1)$ . The received symbol for a *mean proportion* is  $\sqrt{\phantom{x}}$ , as in  $OQ = \sqrt{OP \cdot OL}$ . Here then  $OQ = \sqrt{(-1 \cdot 1)} = \sqrt{-1}$ . This is only a following out of *analogy* with the symbol; though, previously,  $\sqrt{\phantom{x}}$  expressed the mean proportion between *numbers*, or perhaps *lengths*, without cognizance of direction.



Now, our first care must be, to inquire whether  $\sqrt{-}$  as a symbol of direction, has the same properties as when it operates on a pure number.

First, in combining factors, the order is indifferent, as  $ab = ba$ , and  $a \cdot (1) = a = 1 \cdot a$ . We ask, does  $\sqrt{-}$  fulfil this condition? Evidently,  $a \cdot \sqrt{-1} = \sqrt{-1} \cdot a$ , each measuring the length  $a$ , directed along the perpendicular  $OQ$ . Similarly

$$a \cdot \sqrt{-b} = \sqrt{-b} \cdot a = b \sqrt{-1} \cdot a = ba \sqrt{-1}.$$

Next, repeat  $\sqrt{-}$ . We had  $OQ = OL \sqrt{-1}$  or  $\sqrt{-1} \cdot OL$ . Also  $OP = \sqrt{-1} \cdot OQ$ , because  $\angle QOP = 90^\circ$ ,  $\therefore OP = \sqrt{-1} \cdot (\sqrt{-1} \cdot OL)$ . But  $OP = -OL$  or  $-1 \cdot OL$ . Evidently then  $\sqrt{-1} \cdot \sqrt{-1}$  is equivalent to  $-1$ . This further justifies the change of  $\frac{1}{\sqrt{-1}}$  to  $-\sqrt{-1}$ .

5. But a new difficulty arises in *adding* unlike quantities, i.e. in connecting them by  $+$ . If along radii  $m$  and  $n$  we have two lengths  $a_m$  and  $b_n$ , what meaning can we attach to  $a_m + b_n$ ? This urgently needs explanation. It may seem that the symbol  $+$  (*plus*) receives

a new sense.—Now in fact when  $(a + b) = \text{zero}$ , the  $+$  does not strictly mean *addition*; it really expresses a *difference*, not a *sum*; but not to embarrass generalization, we call it a *sum*, and say that either  $a$  or  $b$  is negative. They may mean the very same line  $OL$  estimated in opposite directions, as  $OL$  and  $LO$ . If  $OL$  mean the line *as travelled* from  $O$  to  $L$ , and  $LO$  the same *as travelled* from  $L$  to  $O$ , the statement  $OL + LO = \text{zero}$ , clearly means that the *total result* of such travel is nothing; since the travel *neutralizes itself*. Thus if, instead of saying that the *sum* is zero, which gives only a *numerical* idea, I call *total result* zero, you will gain a *geometrical* idea. At this we must aim, when we deal with lines differing in direction. Evidently, if, starting from any point in the outline of a limited surface, a point travel round the circuit, until it regain its original place, we may justly say, *the total result* of such change is *zero*; and no one will suppose it to mean that the *length of the circuit* is zero. So if there be a triangle  $ABC$ , we may say, the total result of the travel  $AB + BC + CA = 0$ , if it be understood that each line is to be estimated in a different direction. Indeed, suppose the *lengths* of the three sides are  $c, a, b$ , then in the equation  $c_m + a_n + b_p = 0$ , the symbol  $+$  cannot mislead us, though its sense is evidently enlarged from sum to total result.

Again, since  $AB + BC + CA = 0$  and  $CA = -AC$ , when direction is considered, we have  $AB + BC - AC = 0$ , which further justifies  $AB + BC = AC$ . The last is interpretable,—“Motion along two successive sides of a triangle yields *the same total result* as motion along the third side.”

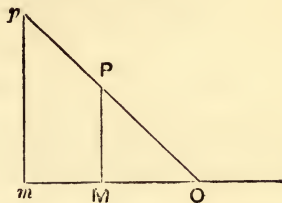
The word resultant characterizes mechanics, but there seems no objection to adopting it in geometry also for the total result, as distinguished from the sum.

6. In fact we have unawares made a great step forward; for the symbol  $\sqrt{-}$  now enables us to express *distance in every direction*. If our parallelogram become a rectangle, and  $AB$  is the ordinary positive axis, and (as before), the *lengths* of  $AB, BC, CA$  are  $c, a, b$ , we have  $BC = a\sqrt{-1}$  when direction is estimated, and  $AC = \text{total result of } c \text{ and } a\sqrt{-1}$ , or  $AC$  (an oblique line)  $= c + a\sqrt{-1}$ . Since  $c$  and  $a$  are independent lengths,  $AC$  may have any direction whatever.

But again, we must inquire whether the symbol  $+$ , thus extended, can be worked in the received method. First, does it fulfil the fundamental condition expressed by  $A + B = B + A$ ? Assuming (as



we must) the doctrine of parallel straight lines, and considering any rectangle whose sides are  $a$  and  $c$ , we find that  $c + a\sqrt{-1}$  = the diagonal  $= a\sqrt{-1} + c$  from the opposite sides. Next, does it fulfil the condition  $h(a + B) = hA + hB$ ? The doctrine of similar triangles at once affirms it. Let  $OM = x$ ,  $MP = y$ , perpendicular to it; join  $OP$ , then if  $OM$  is the positive axis, and  $y$  expresses mere length, we write



$$OP \text{ or } OM + MP = x + y\sqrt{-1}.$$

Next, along  $OM$  take  $Om = hx$ ; that is  $1 : h = x : Om$  (whether  $h$  is linear or numerical). Erect  $mp$  perpendicular to  $Om$  and meeting  $OP$  in  $p$ . Then by similar triangles,  $\overline{mp} = h \cdot y$  (in length) and  $Op = h \cdot OP$ . In this  $h \cdot OP = Op$  we have supposed  $h$  to be numerical.

Also  $OP$  is equivalent to  $x + \sqrt{-1} \cdot y$ ,

and  $Op$  to  $Om + mp\sqrt{-1}$  or  $hx + \sqrt{-1} \cdot hy$ ,

that is,  $h(x + \sqrt{-1}y) = hx + \sqrt{-1} \cdot hy$ ,

just as if  $\sqrt{-1}$  were numerical.

If further we change  $h$  from a mere number to a positive length, it affects every term of the last in the same ratio, and leaves equivalence as before.

If we have proved generally that with any factor  $h$  (provided it be counted along the *positive* axis) the product  $h(x + \sqrt{-1} \cdot y)$  is equivalent to  $hx + \sqrt{-1} \cdot hy$ , the same is virtually proved, if  $h$  be changed into  $h_m$ , that is, if the numerical  $h$  be computed along an  $m$ -axis. For we may transform our hypothesis, by choosing the  $m$ -axis as positive. If hereby  $x, y$  change to  $x', y'$ , we obtain a result the same *in form* as the previous result, and  $x', y'$  remain quite as general as were the  $x, y$ . Thus we may write

$$h_m \cdot (x + \sqrt{-1}y) = h_mx + \sqrt{-1} \cdot h_my.$$

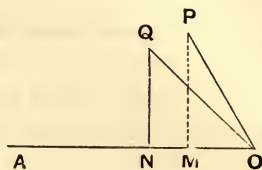
After this, we can change the oblique  $h_m$  into  $a + \sqrt{-1} \cdot b$ , where  $a, b$  are along the positive axis. Now if the  $m$ -axis be perpendicular to the positive, we may write simply  $h_m = k\sqrt{-1}$ , where  $k$  is along the positive axis. Then  $\sqrt{-1} \cdot h_my = \sqrt{-1} \cdot \sqrt{-1} \cdot ky$ , and since each  $\sqrt{-1}$

denotes revolution of the  $ky$  through  $90^\circ$ , the  $\sqrt{-1} \cdot \sqrt{-1}$  shews revolution through  $180^\circ$ , or is equivalent to the symbol  $-$ . Thus

$$\begin{aligned} h_m(x + \sqrt{-1}y) &= \sqrt{-1} \cdot k(x + \sqrt{-1}y) \\ &= \sqrt{-1} \cdot kx + \sqrt{-1} \cdot k \cdot \sqrt{-1} \cdot y \\ &= \sqrt{-1}kx - ky \end{aligned}$$

exactly as if  $\sqrt{-1}$  were numerical. Evidently then the same holds good in multiplying out  $(h + \sqrt{-1}k)$  into  $(x + \sqrt{-1}y)$ .

**THEOREM.** If  $A + B\sqrt{-1} \equiv C + D\sqrt{-1}$ , this implies *two* equalities, viz.  $A = C$  and  $B = D$ . The geometrical proof appeals at once to the eye. If  $OA$  be the positive axis, and the binomials are denoted by  $OP$  and  $OQ$ , viz.  $A + \sqrt{-1}B = OP$  and  $C + \sqrt{-1}D = OQ$ , we do not account  $OP = OQ$  until they have the *same direction*, as well as the same length. This requires  $Q$  to *coincide* with  $P$ . Of course then, if  $PM, QN$  are dropt perpendicular to  $OA$  we have  $OM = A$ ,  $PM = B\sqrt{-1}$ ,  $ON = C$ ,  $QN = D\sqrt{-1}$ , and as soon as  $OP = OQ$  in our hypothesis,  $P$  coincides with  $Q$ , therefore also  $M$  with  $N$ . That is  $A = C$  and  $B = D$ .



The reader will now see the geometrical meaning of the “imaginary roots” (so called) of a quadratic equation. As a very simple case, take first

$$x^2 - 16x + 63 = 0, \text{ which yields } x = 8 \pm 1.$$

Here both roots lie along the positive axis. But change 63 to 65, then

$$x^2 - 16x + 65 = 0, \text{ whence } x = 8 \pm \sqrt{-1}.$$

In the latter the two roots are equal radii drawn from the origin at equal angles on opposite sides, radii which terminate where the coordinate along the axis is 8, and the transverse coordinate is  $\pm 1$ .

# TRACT III.

## ON FACTORIALS.

### SUPPLEMENT II.

#### *Extension of the Binomial Theorem.*

1. THE following appeared in Cauchy's elementary treatise, as early, I think, as 1825, but without the new Factorial Notation, which adds much to its simplicity. Boole writes  $x^{(2)}$  for  $x(x-1)$ ,  $x^{(3)}$  for  $x(x-1)(x-2)$ , and  $x^{(n)}$  for  $x(x-1)(x-2)\dots(x-n+1)$ , whence  $x^{(n+1)} = x^{(n)} \cdot (x-n)$ . Better still it is, to place the exponent in a half-oval, since a parenthesis ought to be *ad libitum*. I propose  $x^{\smile}$ ,  $x^{\smile-1}$  which are quite distinctive. Then the Binomial Theorem is

$$(1+x)^n = 1 + n \cdot \frac{x}{1} + n^{\smile} \cdot \frac{x^2}{1 \cdot 2} + n^{\smile-1} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + x^n.$$

In this notation  $1 \cdot 2 \cdot 3 \cdot 4 \dots n$  or  $n(n-1)\dots 2 \cdot 1$  is  $n^{\smile}$ . Gudermann for this has  $n'$ ; but  $(n-1)'$  is less striking to the eye than  $\underline{n}$ ,  $\underline{n-1}$  introduced by the late Professor Jarrett. This exhibits in the Binomial Theorem its general term, by

$$(1+x)^n = 1 + n \cdot \frac{x}{1} + \dots + n^{\smile-r} \cdot \frac{x^r}{\underline{r}} + \dots + x^n;$$

of course  $x^{\smile-1}$  is equivalent to simple  $x$ .

The *Exponent* (of a power) is already distinguished from an *Index*. In a Factorial  $x^{\smile-r}$  for  $x(x-1)(x-2)\dots(x-r-1)$  one may call  $r$  (which must be integer) the *Numero*, as stating the number of factors.

2. If  $m, n, p$  be all positive integers, and  $p = m + n$ , then

$$(1+x)^m \cdot (1+x)^n = (1+x)^p,$$

or, in condensed expansion,

$$\left\{1 + \Sigma m^r \cdot \frac{x^r}{r} + x^m\right\} \cdot \left\{1 + \Sigma n^r \cdot \frac{x^r}{r} + x^n\right\} = \left\{1 + \Sigma p^r \cdot \frac{x^r}{r} + x^p\right\}.$$

This equation being of the  $(m+n)^{\text{th}}$  or  $p^{\text{th}}$  degree of  $x$ , and being true for values of  $x$  indefinite in number (therefore in more than  $p$  values), must be equal term by term for every power of  $x$ . Now when we multiply any two such series,

$$(1 + M_1x + M_2x^2 + M_3x^3 + \text{etc.}) \text{ by } (1 + N_1x + N_2x^2 + \text{etc.}),$$

we have a product of the same form

$$1 + P_1x + P_2x^2 + P_3x^3 + \dots$$

by the routine of multiplication, in which

$$P_1 = M_1 + N_1; \quad P_2 = M_2 + M_1N_1 + N_2; \quad P_3 = M_3 + M_2N_1 + M_1N_2 + N_3;$$

and the law of the indices is so visible, that we get generally

$$P_r = M_r + M_{r-1}N_1 + M_{r-2}N_2 + \dots + M_1N_{r-1} + N_r.$$

This being true for all series of this form may be applied to the three series  $(1+x)^m$ ,  $(1+x)^n$ ,  $(1+x)^p$ , and at once it yields to us the result

$$\frac{p^r}{r} = \frac{m^r}{r} + \frac{m^{r-1}}{r-1} \cdot \frac{n}{1} + \frac{m^{r-2}}{r-2} \cdot \frac{n^2}{2} + \frac{m^{r-3}}{r-3} \cdot \frac{n^3}{3} + \dots + \frac{n^r}{r};$$

an equation true for more values of  $m$  and  $n$  than are counted by the integer  $r$ , therefore it is true also when  $m$  and  $n$  are arbitrary and fractional. Write  $x$  for  $m$ ,  $h$  for  $n$ , and  $x+h$  for  $p$ , and you have an *extension* of the ordinary  $(x+h)^r$ . For, this latter may be written

$$\frac{(x+h)^r}{r} = \frac{x^r}{r} + \frac{x^{r-1}}{r-1} \cdot \frac{h}{1} + \frac{x^{r-2}}{r-2} \cdot \frac{h^2}{2} + \frac{x^{r-3}}{r-3} \cdot \frac{h^3}{3} + \dots + \frac{h^r}{r};$$

with *exponents* replacing the *numeros* of the preceding.

NOTE. The reader must carefully observe in that which follows, that the upper index of  $P, Q, A, B, C$ , is not an exponent.



*Powers in Series of Factorials.*

3. Since  $x^{\smile} = x(x-1) = x^2 - x$ ,  
of which the general law is

$$x^{\smile r} (x-r) = x^{\smile r+1},$$

conversely

$$x^2 = x^{\smile} + x.$$

Again

$$x^3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x,$$

$$\therefore x^3 = x^{\smile} + 3x^2 - 2x.$$

But

$$3x^2 = 3x^{\smile} + 3x,$$

$$\therefore x^3 = x^{\smile} + 3x^{\smile} + 3x.$$

Evidently we can thus in succession obtain  $x^4, x^5, \dots$  and generally  $x^n$  in series of  $x^{\smile}, x^{\smile-1}, x^{\smile-2}, \dots x^{\smile}, x$ .

Since only  $x^{\smile}$  contains  $x^n$ , its coefficient must be 1. In general, with unknown coefficients  $P$ , dependent on  $n$ , but not involving  $x$ , we may write

$$x^n = P_0^n . x^{\smile} + P_1^{n-1} x^{\smile-1} + P_2^{n-2} x^{\smile-2} + \dots + P_{n-2}^2 x^{\smile} + P_{n-1}^1 x \dots (a).$$

Here the lower index denotes the place of the term in the series; the sum of upper and lower index =  $n$ , the exponent of  $x^n$ ; and the upper index is the same as the *numero* of its term. We have also seen that  $P_0^n = 1$ , whatever  $n$  may be. It remains to calculate  $P_r^{n-r}$ . Multiply the left member of (a) by  $x$ , and on the right multiply the successive terms by the *equivalents* of  $x$ , viz.

$$(x-n) + n, (x-n+1) + (n-1), (x-n+2) + (n-2), \text{ etc.},$$

and apply to each term the formula  $x^{\smile} . (x-r) = x^{\smile+1}$ . Then

$$\left. \begin{aligned} x^{n+1} = & P_0^n . x^{\smile+1} + P_1^{n-1} . x^{\smile} + P_2^{n-2} x^{\smile-1} + \dots + P_{n-2}^2 x^{\smile} + P_{n-1}^1 x^{\smile} \\ & + n . P_0^n . x^{\smile} + (n-1) P_1^{n-1} . x^{\smile-1} + (n-2) P_2^{n-2} x^{\smile-2} + \dots \\ & + 2P_{n-2}^2 x^{\smile} + 1P_{n-1}^1 x \end{aligned} \right\} (b).$$

But if in (a) we write  $(n+1)$  for  $n$ , we have

$$x^{n+1} = P_0^{n+1} x^{\smile+1} + P_1^n x^{\smile} + P_2^{n-1} x^{\smile-1} + \dots + P_{n-1}^2 x^{\smile} + P_n^1 x \dots (c),$$

and we cannot be wrong in *identifying* (b) and (c); that is, in equating the coefficients belonging to every particular *numero* ( $r$ ). At the right hand end  $P_n^1 = 1P_{n-1}^1$ , coefficients of  $x$ ; i.e. since when  $n=2$ ,  $x^2 = x^{\smile} + x$ , so that  $P_1^1 = 1$ ,

$$\therefore P_2^1 = 1, P_3^1 = 1;$$

and universally  $P_n^1 = 1$ , just as  $P_0^n = 1$ . Also in general we find

$$P_{n+1-r}^r = rP_{n-r}^r + P_{n+1-r}^{r-1},$$

otherwise,

$$P_p^r = rP_{p-1}^r + P_p^{r-1},$$

if

$$p = n + 1 - r.$$

This enables us to fill in the vacancies of a table, beginning from

1	1	1	1
1	$P_1^2$	$P_1^3$	$P_1^4$
1	$P_2^2$	$P_2^3$	$P_2^4$
1	$P_3^2$	$P_3^3$	$P_3^4$
1	$P_4^2$	$P_4^3$	$P_4^4$

one horizontal row and one vertical, each consisting of units. Each  $P$  is computed from the  $P$  above it, multiplied by its upper index (which is the number of its column) + its companion to the left in the same row. Thus to form the second row from

1	1	1	1	1
1	$2 \cdot (1) + 1$ = 3	$3 \cdot (1) + 3$ = 6	$4 \cdot (1) + 6$ = 10	$5 \cdot (1) + 10$ = 15

Evidently this second row is

$$1; 1+2; 1+2+3; 1+2+3+4; \dots$$

of which the general term is  $P_1^n = \frac{1}{2}n \cdot (n+1)$ . Similarly from the second row we form the third, working from left to right, and the law is manifest.

1	3	6	10	15	21
1	$2 \cdot (3) + 1$ = 7	$3 \cdot (6) + 7$ = 25	$4 \cdot (10) + 25$ = 65	$5 \cdot (15) + 65$ = 140	$6 \cdot (21) + 140$ = 266

Hence a table to any extent required can be made, such as is here presented.

	$P^1$	$P^2$	$P^3$	$P^4$	$P^5$	$P^6$
0	1	1	1	1	1	1
1	1	3	6	10	15	21
2	1	7	25	65	140	266
3	1	15	90	350	1050	2646
4	1	31	301	1701	6951	
5	1	63	966	7770	42,525	
6	1	127	3025	34,105	246,730	
7	1	255	9330			

When this table is used solely to evaluate the coefficients of  $(a)$ , the indices of  $P_0^n, P_1^{n-1}, P_2^{n-2} \dots$  warn us that the numbers will be taken out *diagonally*. Thus for  $x^6$  we take out (beginning from  $P_0^6$  at the top on right hand) 1, 15, 65, 90, 31, 1.

It will be observed that the *second column* is  $2^1 - 1, 2^2 - 1, 2^3 - 1$  and in general  $P_r^2 = 2^{r+1} - 1$ .

Such is the Table of *Direct Factorials*.

The letter  $P$  being almost appropriated for the Legendrian functions, I see an advantage in superseding  $P_r^n$  by  $\left| \begin{smallmatrix} n \\ r \end{smallmatrix} \right|$ . Necessarily  $n$  and  $r$  are both integers,  $n$  index of the column,  $r+1$  index of the row.

Collect the results for  $P$

$$P_0^n = 1; P_1^n = 1; P_1^n = \frac{1}{2}n(n+1); P_n^2 = 2^{n+1} - 1;$$

and in general 
$$P_p^r = rP_{p-1}^r + P_p^{r-1},$$

which yield identically

$$x^n = \left| \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right| \cdot x^n + \left| \begin{smallmatrix} n-1 \\ 1 \end{smallmatrix} \right| x^{n-1} + \left| \begin{smallmatrix} n-2 \\ 2 \end{smallmatrix} \right| x^{n-2} + \text{etc.}$$

### Factorials in Series of Powers.

4. This is a mere problem of common multiplication,

$$x^n = x(x-1)(x-2) \dots (x-n+1),$$

yet the factors being special and their combinations often recurring,

the work of one computer may avail for many after him. We may assume

$$x^n = Q_0^n x^n - Q_1^{n-1} x^{n-1} + Q_2^{n-2} x^{n-2} - \text{etc.} \dots \pm Q_{n-1}^1 x \dots (a);$$

where, as before, obviously  $Q_0^n = 1$ . Also dividing by  $x$ , and then making  $x = 0$ , you have

$$Q_{n-1}^1 = \pm 1.2.3 \dots (n-1) = \pm \lfloor n-1.$$

Since

$$(x-n) x^n = x^{n+1},$$

multiply (a) by  $x-n$ ,

$$\therefore x^{n+1} = Q_0^n x^{n+1} - Q_1^{n-1} x^n + Q_2^{n-2} x^{n-1} - \text{etc.} \dots \pm Q_{n-1}^1 x^2 - n Q_0^n x^n + n Q_1^{n-1} x^{n-1} - \text{etc.} \dots \pm n Q_{n-2}^2 x^2 \mp n Q_{n-1}^1 x \} (b).$$

Also in (a) write  $n+1$  for  $n$ ;

$$\therefore x^{n+1} = Q_0^{n+1} x^{n+1} - Q_1^n x^n + Q_2^{n-1} x^{n-1} - \text{etc.} \dots \mp Q_n^1 x \dots (c).$$

But (b) and (c) ought to be identical. We have anticipated the remarks that  $Q_0^{n+1} = Q_0^n = 1$ , and  $Q_n^1 = n Q_{n-1}^1$ , inasmuch as

$$Q_n^1 = 1.2.3 \dots n; = \lfloor n.$$

But generally,

$$Q_{r+1}^{n-r} = n Q_r^{n-r} + Q_{r+1}^{n-r-1};$$

or, if  $n-r = m$ ,

$$Q_{r+1}^m = (m+r) Q_r^m + Q_{r+1}^{m-1}.$$

As before, this enables us to continue the table, when the first row and first column are known. To compare our formula with that of the first table, we may write it

$$Q_p^r = (r+p-1) Q_{p-1}^r + Q_p^{r-1}.$$

In fact, the first row is unity as before. The first column is

$$1, 1, 1.2, 1.2.3, 1.2.3.4,$$

when

$$r = 0, Q_1^m = m Q_0^m + Q_1^{m-1},$$

also in the former table, when

$$n = r, P_1^r = r P_0^r + P_1^{r-1}.$$

Hence the second row is the same in the new table as in the old.

To compute the third row from the second:

1	3	6	10	15	21
1.2	4.(3)+1.2 = 11	4.(6)+11 = 35	5.(10)+35 = 85	6.(15)+85 = 175	$Q_p^r = (r+p-1) Q_{p-1}^r + Q_p^{r-1}$



The multiplier  $r + (p - 1)$  combining upper and lower index of its  $Q$  distinguishes the  $Q$  table from the  $P$  table: thus

	11	35	85
1.2.3	$4.(11) + 1.2.3$ $= 50$	$5.(35) + 50$ $= 225$	$6.(85) + 225$ $= 735$
1.2.3.4	$5.(50) + 2.3.4$ $= 274$	$6.(225) + 274$ $= 1624$	etc.

or indeed

$$Q_p^{r+1} = (r + p) Q_{p-1}^{r+1} + Q_p^r.$$

### *Inverse Factorials.*

	$Q^1$	$Q^2$	$Q^3$	$Q^4$	$Q^5$	$Q^6$	$Q^7$
0	1	1	1	1	1	1	1
1	1	3	6	10	15	21	
2	1.2	11	35	85	175		
3	1.2.3	50	225	735			
4	$\underline{4}$	274	1624	6769			
5	$\underline{5}$	1764	13,132				
6	$\underline{6}$						

These too are used diagonally for  $x^n$ . Thus

$$x^7 = x^7 - 21x^6 + 175x^5 - 735x^4 + 1624x^3 - 1764x^2 + \underline{6}.x.$$

Again it seems better to supersede

$$Q_r^n \text{ by } \left[ \begin{array}{c} \diagup \\ \diagdown \end{array} \right]_r, \text{ then } x^n = \left[ \begin{array}{c} \diagup \\ \diagdown \end{array} \right]_0 x^n - \left[ \begin{array}{c} \diagup \\ \diagdown \end{array} \right]_1 x^{n-1} + \left[ \begin{array}{c} \diagup \\ \diagdown \end{array} \right]_2 x^{n-2} + \text{etc.}$$

5. Even in Arithmetic we are driven upon "recurring decimals," and learn that an infinite series may tend to a unique finite limit. Nor can Elementary Algebra fail to recognize, from

$$\frac{1 - x^n}{1 - x} = 1 + x + x^2 + \dots + x^{n-1};$$

that when  $x$  is numerically less than 1, with  $n$  indefinitely increasing, the series  $1 + x + x^2 + \dots + x^n$  tends to the limit  $\frac{1}{1-x}$ .

After this it quickly follows (by Cauchy's process now perhaps universal), from Binomial Theorem with  $n$  positive integer, that  $\left(1 + \frac{1}{n}\right)^n$  with  $n$  infinite, has for limit

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \text{etc.} = 2.7182818\dots$$

which we call  $e$ , and that  $\left(1 + \frac{a}{n}\right)^n$  has for limit

$$1 + \frac{a}{1} + \frac{a^2}{1 \cdot 2} + \frac{a^3}{1 \cdot 2 \cdot 3} + \text{etc.} \dots$$

which also  $= e^a$ . On this we need not here dwell: but it will presently be assumed. Now let us propound

*Factorials with Negative Numero.*

6. Analogy suggests to define  $x^{-2}$  as meaning  $\frac{1}{x(x+1)}$ : of course  $x^{-1}$  will be identical with  $x^{-1}$ . Then  $x^{-3}$  will stand for

$$[x(x+1)(x+2)]^{-1},$$

and  $x^{-n}$  for  $[x(x+1)(x+2)\dots(x+n-1)]^{-1}$ .

Hence 
$$x^{-(n-1)} = (x+n)^{-1} \cdot x^{-n}.$$

Now 
$$x^{-2} = (x+1)x^{-1},$$

and when  $x$  is  $> 1$ , we have, in descending powers

$$x^{-2} = x^{-2} - x^{-3} + x^{-4} - x^{-5} + \text{etc.}$$

Also 
$$x^{-3} = (x+3)^{-1} \cdot x^{-2}$$

of the two factors here on the right, each can take the form of a series descending in powers of  $x$ . So then their product, the equivalent of  $x^{-3}$ . By like reasoning we claim a right to assume with coefficients independent of  $x$ ,

$$x^{-(n-1)} = A_0^n x^{-n-1} - A_1^n x^{-n-2} + A_2^n x^{-n-3} - \text{etc.} \dots\dots\dots (a),$$

and our task is to discover the coefficients when  $n$  is given.

First make  $n = 1$ ,

$$\therefore x^{\smile} = A_0^1 x^{-2} - A_1^1 x^{-3} + A_2^1 x^{-4} - A_3^1 x^{-5} + \text{etc.}$$

But from the series already obtained for  $x^{\smile}$  we see that every coefficient of the last is 1; or in general  $A_r^1 = 1$ . This gives the first *column* of our table. Also universally  $A_0^n$  obviously = 1; which fixes the first *row* of the table.

Next, multiply equation (a) by  $(x + n)$ , and you get

$$\left. \begin{aligned} x^{\smile} &= x^{-n} - A_1^n x^{-n-1} + A_2^n x^{-n-2} - \text{etc.} \dots \pm A_r^n x^{-n-r} \mp \dots \\ &+ n x^{-n-1} - n \cdot A_1^n x^{-n-2} + \text{etc.} \dots \mp n A_{r-1}^n x^{-n-r} \mp \dots \end{aligned} \right\} \dots (b).$$

But in (a) we may write  $n$  for  $n + 1$ , which gives

$$x^{\smile} = x^{-n} - A_1^{n-1} x^{-n-1} + \dots \pm A_r^{n-1} x^{-n-r} \mp \dots (c).$$

Now (b) and (c) must be identical, hence

$$A_1^n = n + A_1^{n-1} = n A_0^n + A_1^{n-1},$$

and generally

$$A_r^n = n A_{r-1}^{n-1} + A_r^{n-1}.$$

But this is exactly the law of  $P$  in Art. 3, only there we had  $r, p$  for what are here  $n, r$ . Now as the first row and first column are here, as there, unity, and the law of continuation the same, the whole table will be the same, and we may write  $P$  of Art. 3 in place of this  $A$ . Thus we get

$$x^{\smile} = x^{-n} - P_1^{n-1} x^{-n-1} + P_2^{n-1} x^{-n-2} - P_3^{n-1} x^{-n-3} + \text{etc.} \dots$$

with the *same* values of  $P$  as before. But in the last equation we no longer take the  $P$ 's *diagonally*, but *vertically*, down the column, as the same upper index  $(n - 1)$  above every  $P$  denotes; thus

$$x^{\smile} = x^{-3} - 3x^{-4} + 7x^{-5} - 15x^{-6} + 31x^{-7} - \text{etc.} \dots$$

To verify, multiply by  $x + 2$ . But for convergence  $x$  ought to exceed 2. So

$$x^{\smile} = x^{-4} - 6x^{-5} + 25x^{-6} - 90x^{-7} + \text{etc.} \dots$$

and for convergence,  $x$  ought to exceed 3. Evidently in the series for  $x^{\smile}$ ,  $x$  ought to exceed  $(n - 1)$ .

7. Assume now the *Inverse Problem*, to develop  $x^{-n}$  in series of Factorials.

With unknown coefficients  $B$  independent of  $x$ , we start from

$$x^{-n-1} = x^{\smile} + B_1^n x^{\smile} + B_2^n x^{\smile} + \dots (a).$$

Multiply the lefthand by  $x$ , and the successive terms on the right by the equivalents of  $x$ , viz.

$$(x+n) - n, (x+n+1) - (n+1), (x+n+2) - (n+2), \text{ etc.}$$

Observe that  $(x+p) x^{(-p-1)} = x^{(-p)};$

$$\left. \begin{aligned} \therefore x^{-n} &= x^{(-n)} + B_1^n \cdot x^{(-n-1)} + \dots + B_r^n \cdot x^{(-n-r)} + \dots \\ -n \cdot x^{(-n-1)} &- (n+1) B_1^n \cdot x^{(-n-2)} - \dots - (n+r) \cdot B_r^n x^{(-n-r-1)} - \dots \end{aligned} \right\} \dots (b).$$

But in (a) we may write  $n$  for  $n+1$ , which gives

$$x^{-n} = x^{(-n)} + B_1^{n-1} x^{(-n-1)} + B_2^{n-1} x^{(-n-2)} + \dots + B_r^{n-1} \cdot x^{(-n-r)} + \dots (c).$$

Identify (b) with (c),

$$\therefore B_1^n = n + B_1^{n-1};$$

and generally  $B_r^n = (n+r-1) B_{r-1}^{n-1} + B_r^{n-1};$

the same formula as for  $Q$  in Art. 4, only  $n$  and  $r$  here standing for what there was  $r$  and  $p$ . Also since  $B_0^n = 1$ , the top row is unity, here as there.

We may further prove that the first column of our  $B$ 's is the same as the first column of the  $Q$ 's. For

$$x^{-2} = x^{(-2)} + B_1^1 x^{(-3)} + B_2^1 x^{(-4)} + \dots \&c.$$

Multiply by  $x$  on the left, also by  $(x+1) - 1$ ,  $(x+2) - 2$ ,  $(x+3) - 3$ , for the successive terms on the right; then

$$\left. \begin{aligned} x^{-1} &= x^{(-1)} + B_1^1 x^{(-2)} + B_2^1 x^{(-3)} + \dots \\ -1 x^{(-2)} &- 2 B_1^1 x^{(-3)} - 3 B_2^1 x^{(-4)} - \dots \end{aligned} \right\}.$$

Obviously  $x^{-1} = x^{(-1)}$ ; and the other terms must annihilate themselves, making  $B_1^1 - 1 = 0$ ;  $B_2^1 - 2B_1^1 = 0$ ;  $B_3^1 - 3B_2^1 = 0$ ; etc.,

or  $B_1^1 = 1, B_2^1 = 2B_1^1 = 1 \cdot 2;$

$$B_3^1 = 3B_2^1 = 1 \cdot 2 \cdot 3, \text{ and } B_n^1 = 1 \cdot 2 \cdot 3 \dots (n-1) n.$$

Thus  $B_2^1 = Q_2^1, B_3^1 = Q_3^1$ , etc. exact. Therefore the  $B$  table is the same as the  $Q$  table *throughout*.

Here also we take the  $Q$ 's vertically, to obtain  $x^{-n}$  in series of factorials as  $x^{-4} = x^{(-4)} + 6x^{(-5)} + 35x^{(-6)} + 225x^{(-7)} + \text{etc.}$

In general  $x^{-n} = x^{(-n)} + Q_1^{n-1} x^{(-n-1)} + Q_2^{n-1} x^{(-n-2)} + \dots + Q_r^{n-1} \cdot x^{(-n-r)} + \text{etc.};$  but special inquiry is needed concerning convergence. Apparently it converges more rapidly than

$$x^{-2} + (x+1)^{-2} + (x+2)^{-2} + \text{etc.}$$



8. To develop  $(\epsilon^x - 1)^n$  or  $\left\{ \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{etc.} \right\}^n$  in powers of  $x^n$ . We may assume for this the form

$$M_n x^n + M_{n+1} x^{n+1} + \dots + M_r x^r + \text{etc.},$$

where  $M_n$  manifestly = 1, and if  $r$  is less than  $n$ ,  $M_r = 0$ . Now by the Binomial Theorem where  $n$  is a positive integer

$$(\epsilon^x - 1)^n = \epsilon^{nx} - \frac{n}{1} \epsilon^{(n-1)x} + \frac{n \cdot 2}{2} \cdot \epsilon^{(n-2)x} - \text{etc.} \pm \frac{n}{1} \epsilon^x \mp 1,$$

out of which we have to pick up the partial coefficients of  $x^r$  which make up  $M_r$  observing that

$$\epsilon^{px} = 1 + \frac{px}{1} + \frac{p^2 x^2}{2} + \frac{p^3 x^3}{3} + \text{etc.}$$

in which we have to make  $p$  successively  $n, n-1, n-2, \dots$  When we make  $p = n$ , the only term that here concerns us is  $\frac{n^r x^r}{r}$ . When

$p = n-1$ , we get  $-\frac{n}{1} \cdot \frac{(n-1)^r \cdot x^r}{r}$ . When  $p = n-2$ , we have

$$+ \frac{n \cdot 2}{2} \cdot \frac{(n-2)^r \cdot x^r}{r},$$

and so on. All have  $r$  in the denominator. Hence

$$M_r \cdot r = n^r - \frac{n}{1} (n-1)^r + \frac{n \cdot 2}{2} \cdot (n-2)^r - \text{etc.} \dots \text{to } n \text{ terms,}$$

of which the last is  $\pm \frac{n}{1} \cdot 1^2$ .

Let  $N$  be to  $(n+1)$  what  $M$  is to  $n$ ; then, with  $r$  the same for both,

$$N_r \cdot r = (n+1)^r - \frac{n+1}{1} \cdot (n)^r$$

$$+ \frac{(n+1) \cdot 2}{2} (n-1)^r - \frac{(n+1) \cdot 3}{3} \cdot (n-2)^r + \dots \text{to } (n+1) \text{ terms.}$$

Add this to the preceding, coupling every pair that with the same value of  $p$  has  $(n-p)^r$  as factor;

$$\begin{aligned} \therefore r \cdot (M_r + N_r) &= (n+1)^r - \left[ \frac{n+1}{1} - 1 \right] (n)^r + \left[ \frac{n+1}{2} - 1 \right] \cdot \frac{n}{1} \cdot (n-1)^r \\ &\quad - \left[ \frac{n+1}{3} - 1 \right] \frac{n \cdot n-1}{1 \cdot 2} (n-2)^r \\ &\quad + \left[ \frac{n+1}{4} - 1 \right] \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} (n-3)^r - \text{etc.} \end{aligned}$$

Observe that  $\frac{n+1}{1} - 1 = n$ ,  $\frac{n+1}{2} - 1 = \frac{n-1}{2}$ ,  $\frac{n+1}{3} - 1 = \frac{n-2}{3}$ ,  
and so on ;

$$\therefore [r \cdot (M_r + N_r) = (n+1)^n - \frac{n}{1} (n)^r + \frac{n \cdot n - 1}{1 \cdot 2} (n-1)^r - \text{etc.},$$

or, in shorter notation,

$$= (n+1)^r - \frac{n}{1} \cdot (n)^r + \frac{n \cdot 2}{2} \cdot (n-1)^r - \frac{n \cdot 3}{3} \cdot (n-2)^2 + \text{etc. to } (n+1) \text{ terms.}$$

But in  $N_r$  change  $r$  to  $r+1$  and you have

$$\begin{aligned} [r+1 \cdot N_{r+1} &= (n+1)^{r+1} - \frac{n+1}{1} (n)^{r+1} + \frac{n+1 \cdot n}{1 \cdot 2} (n-1)^{r+1} \\ &\quad - \frac{n+1 \cdot n \cdot n - 1}{1 \cdot 2 \cdot 3} \cdot (n-2)^{r+2} + \text{etc. ... to } (n+1) \text{ terms} \\ &= (n+1) \left\{ (n+1)^r - \frac{1}{1} (n)^{r+1} + \frac{n}{1 \cdot 2} (n-1)^{r+1} - \frac{n \cdot n - 1}{1 \cdot 2 \cdot 3} (n-2)^{r+1} + \text{etc.} \dots \right\} \\ &= (n+1) \left\{ (n+1)^r - \frac{n}{1} (n)^r + \frac{n \cdot n - 1}{1 \cdot 2} (n-1)^r \right. \\ &\quad \left. - \frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} (n-2)^r + \text{etc.} \right\} \\ &= (n+1) \cdot [r (M_r + N_r), \text{ from above. Divide by } (n+1) \cdot [r+1 ; \\ &\therefore \frac{N_{r+1}}{n+1} = \frac{M_r + N_r}{r+1} \dots\dots\dots (a). \end{aligned}$$

Our  $r$  always exceeds  $n$ . We simplify the notation somewhat by assuming as coefficients after dividing by  $x^n$ ,

$$\begin{aligned} \left( \frac{e^x - 1}{x} \right)^n &= 1 + \frac{C_1^n x}{n+1} + \frac{C_2^n x^2}{n+1 \cdot n+2} + \frac{C_3^n x^3}{n+1 \cdot n+2n+3} + \dots \\ &\quad + \frac{C_p^n x^p}{n+1 \cdot n+2 \dots n+p} + \text{etc.} \end{aligned}$$

Multiply by  $x^n$ . The general term becomes

$$\frac{C_p^n \cdot x^{n+p}}{(n+1)(n+2) \dots (n+p-1)(n+p)}.$$

Put  $n + p = r$ , to identify this term with our previous  $M_r \cdot x^r$ ;

$$\therefore M_r = \frac{C_{r-n}^n}{n+1 \cdot n+2 \cdot n+3 \dots (r-1) \cdot r};$$

so that  $N_r = \frac{C_{r-n-1}^{n+1}}{n+2 \cdot n+3 \dots r-1 \cdot r}$  with one less factor in denominator: but when both  $n$  and  $r$  in  $M_r$  are increased by 1,  $r-n$  undergoes no change; and  $N_{r+1} = \frac{C_{r-n}^{n+1}}{n+2 \cdot n+3 \dots r-1 \cdot r \cdot r+1}$ .

Introduce these values into (a) and multiply by

$$(n+1)(n+2) \dots r \cdot (r+1), \therefore C_{r-n}^{n+1} = C_{r-n}^n + (n+1) \cdot C_{r-n-1}^{n+1} \dots (b).$$

Change  $r-n$  to  $p$  and  $n+1$  to  $n$ ,  $\therefore C_p^n = C_p^{n-1} + nC_{p-1}^n$ ; the same law as for the  $P$ 's in Art. 3.

Also when  $n = 1$ ,

$$\left(\frac{\epsilon^x - 1}{x}\right)^1 = 1 + \frac{x}{1 \cdot 2} + \frac{x^2}{1 \cdot 2 \cdot 3} + \dots + \frac{x^r}{1 \cdot 2 \dots (r+1)} + \text{etc.}$$

But we assumed as equivalent

$$1 + \frac{C_1^1 x}{2} + \frac{C_2^1 x^2}{2 \cdot 3} + \frac{C_3^1 x^3}{2 \cdot 3 \cdot 4} + \dots + \frac{C_r^1 x^r}{2 \cdot 3 \dots (r+1)} + \dots$$

which identifies  $C_r^1$  with 1, for all values of  $r$ . Just so  $P_r^1 = 1$  in its first column. Also evidently  $C_0^n = 1$ ; just as  $P_0^n = 1$ , in the whole first row. Thus, with first column and first row identical with those of  $P$  and the same law of continuation, the whole table is the same. Finally then we obtain

$$\left(\frac{\epsilon^x - 1}{x}\right)^n = 1 + \frac{P_1^n \cdot x}{n+1} + \frac{P_2^n \cdot x^2}{n+1 \cdot n+2} + \frac{P_3^n \cdot x^3}{n+1 \cdot n+2 \cdot n+3} + \text{etc.} \dots (c)$$

in which the increasing numerators are pulled down by increasing denominators.

The general term may be written  $n^{\text{p}} \cdot P_p^n \cdot x^p$ , or equally

$$\frac{P_p^n \cdot x^p}{(n+p)^{\text{p}}}.$$

[The course of analysis here pursued forestalls that of  $\Delta^n \cdot 0^r$  to the learner.]

Thus

$$\left(\frac{\epsilon^x - 1}{x}\right)^n = 1 + \frac{\begin{array}{|c|} \hline \text{1} \\ \hline \end{array} x}{n+1} + \frac{\begin{array}{|c|} \hline \text{2} \\ \hline \end{array} x^2}{n+1 \cdot n+2} + \frac{\begin{array}{|c|} \hline \text{3} \\ \hline \end{array} x^3}{n+1 \cdot n+2 \cdot n+3} + \text{etc.}$$

9. To investigate  $\log(1+x)$  in series, with no aid from  $(1+x)^n$  except in the elementary case of  $n$  being a positive integer, and no aid from the Higher Calculus.

Let  $y = e^x - 1$ , then  $x = \log(1+y)$ . Also

$$y = \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \text{etc.} \dots$$

by elementary algebra.

From the last we see that for minute values of  $x$ , as a *first* approximation,  $y = x$ , and  $x^2 = y^2$ . As a second,  $y = x + \frac{1}{2}x^2 = x + \frac{1}{2}y^2$ ,  $\therefore x = y - \frac{1}{2}y^2$ . *Whatever* the series in powers of  $x$ ,  $x$  can be thus *reverted* into powers of  $y$ , at least *within certain limits*. Hence we may assume with unknown coefficients called  $A$ ,

$$x \text{ or } \log(1+y) = y - A_2 y^2 + A_3 y^3 - A_4 y^4 + \text{etc.} \dots (a),$$

which (with the appropriate numerical values of  $A_2, A_3, A_4, \dots$ ) will be an *identical* equation.

Consequently, writing  $y+z$  for  $y$ ,

$$\log(1+y+z) = (y+z) - A_2(y+z)^2 + A_3(y+z)^3 + \text{etc.} \dots (b).$$

Subtract (a) from (b) developing the powers

$$(y+z)^2, (y+z)^3, (y+z)^4, \dots$$

then  $\log(1+y+z) - \log(1+y) = z - A_2(2yz + z^2)$

$$+ A_3(3y^2z + 3yz^2 + z^3) - A_4(4y^3z + 6y^2z^2 + 4yz^3 + z^4) + \text{etc.} \dots (c).$$

But the left hand =  $\log \frac{1+y+z}{1+y}$  or  $\log \left(1 + \frac{z}{1+y}\right)$  which again,

by (a), if we write  $\frac{z}{1+y}$  for  $y$ , has for equivalent

$$\left(\frac{z}{1+y}\right) - A_2 \cdot \left(\frac{z}{1+y}\right)^2 + A_3 \left(\frac{z}{1+y}\right)^3 - \text{etc.} \dots$$

of which the first term alone contains the simple power of  $z$ , and there its whole coefficient is  $\left(\frac{1}{1+y}\right)$ . This then must be the sum of the partial coefficients of  $z$  found in (c).

$$\text{That is } \frac{1}{1+y} = 1 - 2A_2y + 3A_3y^2 - 4A_4y^3 + \text{etc.}$$

But  $\frac{1}{1+y} = 1 - y + y^2 - y^3 + \text{etc.}$  when  $y$  is numerically less than 1.



Hence  $2A_2=1$ ,  $3A_3=1$ ,  $4A_4=1$ , ...  $nA_n=1$ , or in general  $A_n=\frac{1}{n}$ .

Finally then,  $\log(1+y)=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\text{etc.}\dots$  while  $y^2<1$ .

N.B. This furnishes the means of computing logarithms in Elementary Algebra, before knowing the Binomial Theorem with negative or fractional exponent. If Differentiation or "Derivation," as far as  $\phi(x)=x^n$ ,  $\phi'(x)=nx^{n-1}$ , be admitted, this equation at once proves that when  $\phi(z)$  means  $\log(z)$ ,  $\phi'(z)=\frac{1}{z}$ : a process which eases the next Article.

10. PROBLEM. To develop  $[\log \overline{1+x}]^n$  in a series of powers of  $x$ .

That this is possible, when  $x^2$  is  $<1$ , the preceding Article shows. We may then assume, with unknown coefficients  $\lambda_1^n, \lambda_2^n, \lambda_3^n \dots$  depending on  $n$  alone, where the upper index is *not* an exponent,

$$z = (\log \overline{1+x})^n = x^n - \frac{\lambda_1^n x^{n+1}}{n+1} + \frac{\lambda_2^n x^{n+2}}{n+1 \cdot n+2} - \text{etc.} \dots$$

$$\pm \frac{\lambda_r^n \cdot x^{n+r}}{n+1 \cdot n+2 \dots n+r} \mp \text{etc.} \dots (a).$$

If  $\log \overline{1+x}=u$ ,  $z=u^n$ ,  $dz=nu^{n-1}du$ , and  $du=\frac{dx}{1+x}$ . [It is hardly worth while to *disguise* Differentials by a more elaborate and tedious Algebra.]

Differentiate both sides of (a), then drop the common factor  $dx$ :

hereby, 
$$\frac{n(\log \overline{1+x})^{n-1}}{1+x} = nx^{n-1} - \lambda_1^n \cdot x^n + \frac{\lambda_2^n x^{n+1}}{n+1} - \text{etc.} \dots$$

$$\pm \frac{\lambda_r^n \cdot x^{n+r-1}}{n+1 \cdot n+2 \dots (n+r-1)} \mp \text{etc.}$$

Multiply by  $1+x$ , and divide by  $n$ ,

$$\therefore (\log \overline{1+x})^{n-1} = x^{n-1} - \lambda_1^n \cdot \frac{x^n}{n} + \frac{\lambda_2^n \cdot x^{n+1}}{n \cdot n+1} - \text{etc.}$$

$$\pm \frac{\lambda_r^n \cdot x^{n+r-1}}{n \cdot n+1 \dots (n+r-1)} \mp \text{etc.} \dots (b).$$

$$+ x^n - \frac{\lambda_1^n \cdot x^{n+1}}{n} + \text{etc.} \mp \frac{\lambda_{r-1}^n \cdot x^{n+r-1}}{n \cdot n+1 \dots (n+r-2)} \pm \text{etc.}$$

But writing  $(n-1)$  for  $n$  in equation (a) we get

$$(\log \overline{1+x})^{n-1} = x^{n-1} - \frac{\lambda_1^{n-1} \cdot x^n}{n} + \frac{\lambda_2^{n-1} \cdot x^{n+1}}{n \cdot n+1} - \text{etc.} \dots$$

$$\pm \frac{\lambda_r^{n-1} \cdot x^{n+r-1}}{n \cdot n+1 \dots (n+r-1)} \mp \text{etc.} \dots (c).$$

Identifying (b) with (c), we obtain  $\lambda_1^n = n + \lambda_1^{n-1}$ ; and generally  $\lambda_r^n = (n+r-1) \lambda_{r-1}^n + \lambda_r^{n-1}$  or  $\lambda_{r+1}^n = (n+r) \lambda_r^n + \lambda_{r+1}^{n-1}$ , the same law as of  $Q$  in Art. 4. Also evidently  $\lambda_0^n = 1$  whatever  $n$  may be, as  $Q_0^n = 1$ . Thus the top row is 1. Again by (a) making  $n=1$ ,

$$\log(1+x) = x - \frac{\lambda_1^1 x^2}{2} + \frac{\lambda_2^1 x^3}{2 \cdot 3} - \frac{\lambda_3^1 x^4}{2 \cdot 3 \cdot 4} + \text{etc.}$$

But this is known to be  $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{etc.}$ , whence

$$\lambda_1^1 = 1, \lambda_2^1 = 2, \lambda_3^1 = 2 \cdot 3, \lambda_4^1 = 2 \cdot 3 \cdot 4, \dots \text{ just as } Q_n^1 = [n].$$

Thus the first column also of  $\lambda$  agrees with that of  $Q$ . In short then, the two tables are the same. Finally:

$$(\log \overline{1+x})^n = x^n - \frac{Q_1^n \cdot x^{n+1}}{n+1} + \frac{Q_2^n \cdot x^{n+2}}{n+1 \cdot n+2} - \frac{Q_3^n \cdot x^{n+3}}{n+1 \cdot n+2 \cdot n+3} + \text{etc.},$$

with coefficients already known.

The analogy to the series of Art. 8 deserves notice.

$$\left( \frac{\log \overline{1+x}}{x} \right)^n = 1 - \frac{\left| \begin{array}{c} n \\ 1 \end{array} \right| x}{n+1} + \frac{\left| \begin{array}{c} n \\ 2 \end{array} \right| x^2}{n+1 \cdot n+2} - \frac{\left| \begin{array}{c} n \\ 3 \end{array} \right| x^3}{n+1 \cdot n+2 \cdot n+3} + \text{etc.}$$

## TRACT IV.

### ON SUPERLINEARS.

SOME *Apology* may seem needful from me, since Dr Todhunter in his volume on *Higher Algebraic Equations* has treated the same subject under the name of "Determinants." Of course I do not pretend to add to him, nor indeed he to Mr Spottiswoode, who carries off all merit on this subject. I read the details with much admiration as treated by the latter, but found his notation by *accents* very dazzling to the eye, in so much as to make it hard to know by sweeping over half a page, what was the meaning of the formula presented to one. Also I found the chief strain in argument and chief liability to error, to turn upon the question, whether this or that resulting term would require a *plus* or a *minus*. By reasoning from linear functions in my own way, though less direct, I was able to avoid this danger and lessen fatigue to my brain. When I exchanged words with my then colleague in University College, London, the late Professor De Morgan, on the question, *why* this topic was not admitted into common Algebra, he replied, that it was *too difficult* for beginners. I have since thought that he might not have so judged, if some of the arguments were otherwise treated; and in fact I have found, that some whom I supposed to speak with authority thought my slight change of method easier to learners. At the same time I must add, that on the very rare occasions in which I have tried to teach an elementary class of mathematics, a mode of reasoning which to one pupil was easier, to another seemed less satisfactory. Perhaps every teacher ought to have "two strings to his bow."

1. Equations of the first degree are called simple, but when three or more letters ( $x, y, z, \dots$ ) are involved, complexity arises with much danger of error, even when there is no difficulty of principle. To solve two equations of the form  $a_1x + b_1y = c_1$ ,  $a_2x + b_2y = c_2$  is *always the same process*. If we could always certainly *remember* the solutions

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_2a_1 - c_1a_2}{a_1b_2 - a_2b_1},$$

and never confound the indices, nor mistake between + and -, this alone might have much value. The modern method, due eminently to the genius of William Spottiswoode, is quite adapted to Elementary Algebra; but its vast range of utility cannot there be guessed.

First study the denominator  $D = a_1b_2 - a_2b_1$ . It arises from the left-hand of the equation  $\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2 \end{cases}$  in which the four letters stand in square, as  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ . Here  $D$  is the difference of the two

products *formed diagonally*, and the diagonal which slopes downward *from left to right* is accepted as the *positive* diagonal. This is a cardinal point. Remember it, and you will not go wrong on + and -. Understand then, that in square with vertical sides

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} \text{ means } MQ - PN.$$

Of course then, so does  $\begin{vmatrix} M & P \\ N & Q \end{vmatrix}$ , which exchanges rows into columns.

But if you change the order of the rows, or the order of the columns, into  $\begin{vmatrix} P & Q \\ M & N \end{vmatrix}$  or  $\begin{vmatrix} N & M \\ Q & P \end{vmatrix}$  you change the result to  $PN - NQ$ , i.e. you change  $D$  to  $-D$ .

After this is fixed in the mind, it is easy to remember the *common denominator* above, viz.  $a_1b_2 - a_2b_1$ , in the form  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ . Call it  $C$ . Then the *numerator* of  $x$  is obtained from  $C$  by changing the column  $a_1a_2$  into the column  $c_1c_2$ , making  $A = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ .

So  $B$  for the numerator of  $y$  is found by changing the column  $b_1b_2$  into  $c_1c_2$ , yielding  $B = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ .

Finally,  $x = \frac{A}{C}$ ,  $y = \frac{B}{C}$ , without mistake.



Observe, if the equations be presented in the form

$$a_1x + b_1y + c_1 = 0; \quad a_2x + b_2y + c_2 = 0;$$

this is equivalent to changing the signs of  $c_1$  and  $c_2$ , which does not affect  $C$ , but exactly reverses the signs of  $A$  and  $B$ . Previously we had

$$(x : A) = (1 : C) = (y : B),$$

$$\text{or } (x : y : 1) = (A : B : C) = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} : \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

But from the two new equations  $a_1x + b_1y + c_1z = 0$   
 $a_2x + b_2y + c_2z = 0$  } you get

$$x : y : z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

in circular order.

We may also present the solution as follows:

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

2. Simple equations are often called *Linear*, by a geometrical metaphor. If a quantity  $u$  is so dependent on  $x, y, z, \dots$  that however the values of these may vary, yet always  $u = ax + by + cz + \dots$  (where  $a, b, c, \dots$  are *numerical*), then  $u$  is called a *linear function* of  $x, y, z, \dots$  its constituents. [One might have expected  $u$  to be called a *dependent* or a *resultant*: but for mysterious reasons of their own, the French have adopted the strange word *function*; and it cannot now be altered.] It is convenient now to set forth a few properties of linear functions. We here suppose the function to have no absolute (constant) term.

I. To multiply every constituent by any number ( $m$ ), multiplies the function by that number.

[For, if  $u = ax + by + cz + \dots$ , then

$$mu = amx + bmy + cmz + \dots]$$

II. If two linear functions have the same number of constituents and these have the same coefficients [as  $U = ax + by + cz$ ,

$U_1 = ax_1 + by_1 + cz_1$ ]; you will add the functions if you join the constituents in pairs [for here

$$U + U_1 = a(x + x_1) + b(y + y_1) + c(z + z_1);$$

whatever the number of constituents].

3. Observing now that  $\begin{vmatrix} a & M \\ b & N \end{vmatrix}$  or  $aN - bM$  is a linear function of  $N$  and  $M$ , we see that to multiply a *column*  $MN$  by  $m$  multiplies the function by  $m$ . Or  $\begin{vmatrix} a, & mM \\ b, & mN \end{vmatrix} = m \begin{vmatrix} a & M \\ b & N \end{vmatrix}$ .

The same is true if we multiply a *row*; for we may regard  $a$  and  $M$  as the variables and  $b$  and  $N$  as constant; then

$$\begin{vmatrix} ma, & mM \\ b, & N \end{vmatrix} = m \begin{vmatrix} a & M \\ b & N \end{vmatrix}.$$

This leads to the remark, that our function ought to be called *superlinear* rather than linear; for it is open to us to suppose constituents alternately constant or variable.

4. Next, by making a column or row binomial, we can sometimes blend two superlinear tablets into one. Thus

$$\begin{vmatrix} A & x \\ B & y \end{vmatrix} + \begin{vmatrix} C & x \\ D & y \end{vmatrix},$$

having the second column the same, yield,

$$(Ay - Bx) + (Cy - Dx) = (A + C)y - (B + D)x = \begin{vmatrix} A + C, & x \\ B + D, & y \end{vmatrix}.$$

Here the column which was in both, remains as before, but the other columns are added and make a binomial. Evidently the same process holds, if a *row*, instead of a column, is the same in both.

Conversely, when a *given* column is binomial, we can resolve the tablet into *two* tablets; and when each column is binomial, we can resolve the tablets into four.

$$\text{Thus } \begin{vmatrix} A + x, & C + z \\ B + y, & D + v \end{vmatrix} = \begin{vmatrix} A, & C + z \\ B, & D + v \end{vmatrix} + \begin{vmatrix} x, & C + z \\ y, & D + v \end{vmatrix}$$

by a *first* process. By a second, each of these tablets becomes two, giving as result,

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} + \begin{vmatrix} A & z \\ B & v \end{vmatrix} + \begin{vmatrix} x & C \\ y & D \end{vmatrix} + \begin{vmatrix} x & z \\ y & v \end{vmatrix}.$$

5. THEOREM. If one column (or row) is *identical with* the other, the tablet = zero. For obviously  $\begin{vmatrix} A & A \\ B & B \end{vmatrix}$ , by definition, is  $AB - BA$  or zero.

COR. Equally the tablet = zero, if one column (or row) be *proportional* to the other. Thus if  $A : B = C : D$ , this proportion yields  $AD = BC$ ; hence

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} \text{ or } \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \text{ meaning } AD - BC, \text{ vanishes.}$$

6. This sometimes usefully simplifies a tablet. Thus

$$\begin{vmatrix} A \pm mC, & C \\ B \pm mD, & D \end{vmatrix}$$

is resolvable by Art. 4 into  $\begin{vmatrix} A & C \\ B & D \end{vmatrix} \pm \begin{vmatrix} mC, & C \\ mD, & D \end{vmatrix}$ . But the second tablet is zero, because its two columns are proportional. Hence the given tablet simply  $\equiv \begin{vmatrix} A & C \\ B & D \end{vmatrix}$ .

COR. Hence an important inference. The value of a tablet is not changed if to one column (or row) you make addition proportional to the other column (or row); nor again if you subtract instead of adding.

Further, if in the original equations of Art. 1, the absolute terms  $c_1, c_2$  become zero, the solution for  $x$  and  $y$  is simply  $x = 0$  and  $y = 0$ . This is indeed the only general solution, unless also the denominator vanish; which makes  $x = \frac{0}{0}, y = \frac{0}{0}$ ; deciding nothing as to the value of  $x$  or  $y$ . In that case we can equate the two values of  $\frac{y}{x}$ ; viz.  $\frac{a_1}{b_1} = -\frac{y}{x} = \frac{a_2}{b_2}$ . Conversely, this shows  $a_1 b_2 \equiv a_2 b_1$ , equivalent to  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ . The last is the *condition* which provides that the two equations shall be mutually *consistent*, though  $x$  and  $y$  do not vanish. It results *from eliminating*  $x$  and  $y$ , leaving their value arbitrary.

7. Begin from the problem of three simple equations,

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}.$$

These present three equations to be fulfilled by only two disposable quantities, viz.  $\frac{x}{z}$  and  $\frac{y}{x}$ , if they are all divided by  $z$ . The three equations are not certainly self-consistent. If  $x, y, z$  be *eliminated*, an equation of condition will remain. Our first business is to investigate it.

From the *two first* equations treated as at the end of Art. 1, but abandoning *circular order*, we have

$$x : y : z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

in which  $-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$  now replaces  $+\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$  of Art. 1.

In the third given equation, substitute for  $xyz$  the three quantities now proved proportional to them, and you get

$$(1) \quad a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

Call it  $V_3 = 0$ . Then  $V_3$  is linear in  $a_3b_3c_3$ . This is the condition that the three equations shall be compatible. It is seen to result from eliminating  $x, y, z$ . Professor De Morgan wished to call these tablets *Eliminants*. Why Gauss entitled them Determinants, no one explains. Spottiswoode apparently introduced the excellent notation

$$(2) \quad V_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \begin{array}{l} \text{Then } V_3 \text{ is linear in } a_3, b_3, c_3, \text{ that is,} \\ \text{in its third row. Hence also in any row.} \\ V_3 \text{ is superlinear of the third order.} \end{array}$$

for the value written above. The coefficient of  $a_3$  is obtained in (1) by obliterating the constituents as last written that are in the same *column* or row as  $a_3$ , thus reducing  $V_3$  to  $\begin{vmatrix} \dots & b_1 & c_1 \\ \dots & b_2 & c_2 \\ a_3 & \dots & \dots \end{vmatrix}$ :

then we see the tablet by which  $a_3$  must be multiplied.



The same process is used with  $b_3$  and  $c_3$ , producing

$$\begin{vmatrix} a_1 & \dots & c_1 \\ a_2 & \dots & c_2 \\ \dots & b_3 & \dots \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & b_1 & \dots \\ a_2 & b_2 & \dots \\ \dots & \dots & c_3 \end{vmatrix}.$$

Finally the signs of the terms of  $V_3$  are alternate

$$+ - +,$$

just as in the bilinear  $V_2 = a_1 b_2 - a_2 b_1$ .

Evidently  $V_3$  is formed of six terms, three positive and three negative; each term having three factors, but in no term is any factor combined with another of its own row or its own column.

In  $c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$  the term  $a_1 b_2 c_3$  (the *diagonal* sloping down from left to right) is positive as before. Thus when  $V_3$  is given in contracted form, we can expand it into three apparent binomials.

8. In the three given equations you may exchange the position of  $x$  and  $y$ ; then by eliminating  $y$ ,  $x$ ,  $z$  you obtain

$$U_3 = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = 0;$$

but you cannot infer that  $U_3 = V_3$ . In fact, to exchange the  $a$  column with the  $b$  column reverses the sign of  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ . Thus it changes  $V_3$  into

$$b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}.$$

That is, to exchange the first and second columns just *reverses the sign* of  $V_3$ . The same effect follows from exchanging any two contiguous columns or rows. Thus generally

$$\begin{vmatrix} A & D & G \\ B & E & H \\ C & F & J \end{vmatrix} = - \begin{vmatrix} D & A & G \\ E & H & B \\ F & C & J \end{vmatrix} = + \begin{vmatrix} D & G & A \\ E & H & B \\ F & J & C \end{vmatrix}.$$

Again, 
$$\begin{vmatrix} A & D & G \\ B & E & H \\ C & F & J \end{vmatrix} = - \begin{vmatrix} B & E & H \\ A & D & G \\ C & F & J \end{vmatrix} = + \begin{vmatrix} B & E & H \\ C & F & J \\ A & D & G \end{vmatrix}.$$

Observe, that if three binomials of  $V_3$  are expressed by

$$ma_3 + na_2 + pa_1$$

the multipliers  $m, n, p$  contain nothing of the column  $a_1, a_2, a_3$ , therefore  $V_3$  is linear in these three constituents. Evidently it is linear in regard to any column; as we before saw, as to any row.

9. THEOREM. Further, *I say*, To exchange rows and columns does not alter the value of  $V_3$ . In proof, multiply the three equations given in Art. 7, by disposable numbers  $m, n, p$ , and so assume  $m, n, p$  that when the three products are added together the coefficients of  $y$  and  $z$  may vanish. There will remain  $(ma_1 + na_2 + pa_3)x = 0$ , and as we do not admit  $x=0$ , we have three equations connecting  $m, n, p$ ; viz.

$$\left. \begin{aligned} a_1m + a_2n + a_3p &= 0 \\ b_1m + b_2n + b_3p &= 0 \\ c_1m + c_2n + c_3p &= 0 \end{aligned} \right\},$$

and that these may be compatible, we need

$$S_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0,$$

by eliminating  $m, n, p$ .

Here  $S_3$  is nothing but  $V_3$  with rows changed to columns and columns to rows, retaining the same positive diagonal  $a_1b_2c_3$ . Every learner will easily find by developing  $V_3$  and  $S_3$  that they are identical: but there is an advantage here in general argument applicable to higher orders.  $S_3=0$  and  $V_3=0$ , being each a condition of compatibility of the previous equations, must contain *the same relation of the constituents*.  $S_3$  is a linear function of its column  $a_1b_1c_1$ ; so is  $V_3$  a linear function of its row  $a_1b_1c_1$ . But  $S_3=0, V_3=0$  being derivable one from the other, there is no possible relation but  $S_3=\mu V_3$  in which  $\mu$  must be free from  $a_1b_1c_1$ . But the same arguments will prove that  $\mu$  is free from  $a_2b_2c_2$ , also from  $a_3b_3c_3$ . Therefore  $\mu$  is *wholly numerical*. Make  $b_1=0, c_1=0$  and this will not affect  $\mu$ . But this makes

$$V_3 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \text{ and } S_3 = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}.$$

But that the two minor tablets are identical was implied in their definition. Hence, on this assumption for  $b_1$  and  $c_1$ , we find  $V_3=S_3$  or  $\mu=1$ . This then is the *universal* value of  $\mu_1$  or  $V_3=S_3$  in all cases. Q. E. D.

10. In the developed value of  $V_3$  (Art. 7) if  $b_3 = 0$  and  $c_3 = 0$ , you get simply  $V_3 = a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$  which does not contain  $a_1$  or  $a_2$ . These

two constituents are made *wholly inefficient* by the vanishing of  $b_3$  and  $c_3$ , and may be changed to zero. Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & 0 & 0 \end{vmatrix}. \text{ So } \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix}.$$

When a major square is divided into two minor squares and two complemental rectangles, the vanishing of *one* rectangle obliterates the other, and the greatest tablet has only the two squares for its factors.

11. Since  $V_3$  is a linear function of every row and of every column, we can argue as in the Second Order or Bilinears, that to multiply any column, or any row, by  $m$ , multiplies  $V_3$  by  $m$ ; and if a column or row consist of Binomials, we resolve the tablet into two. Conversely two  $V_3$ s which differ only in a single row or single column can be joined into one universally,

$$\begin{vmatrix} A + m, & A' & A_0 \\ B + n, & B' & B_0 \\ C + p, & C' & C_0 \end{vmatrix} = \begin{vmatrix} A & A' & A_0 \\ B & B' & B_0 \\ C & C' & C_0 \end{vmatrix} + \begin{vmatrix} m & A' & A_0 \\ n & B' & B_0 \\ p & C' & C_0 \end{vmatrix}.$$

12. Further, if two contiguous columns (or rows) be identical, the  $V_3 = 0$ . For to exchange them changes  $V_3$  to  $-V_3$ . Yet the exchange leaves  $V_3$  exactly what it was before. Hence  $V_3 = -V_3$ . This can only be when  $V_3 = 0$ .

It follows that if a column (or row) be *proportional* to a contiguous column (or row), the tablet is zero. For instance, if  $a_1, b_1, c_1$  are proportional to  $a_2, b_2, c_2$ , we may assume  $a_1 = ma_2, b_1 = mb_2, c_1 = mc_2$ , then

$$V_3 = \begin{vmatrix} ma_2 & mb_2 & mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This gives

$$V_3 = m \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and the last tablet is zero, because its first and second row are identical.

What has been said in this Article of two *contiguous* rows or columns is evidently true of *any two* rows or any two columns, since exchange of contiguous rows (or columns) does but multiply the tablet by  $-1$ .

13. The same argument as before in the Second Order now proves that a tablet  $V_3$  is not changed in value, if any row (or column) receives increase or decrease proportional to some other row (or column). Thus we have shown in  $V_3$  the same properties as those enunciated in  $V_2$ .

14. NEW PROBLEM: to solve for  $x, y, z$  in the three equations

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0, \\ a_3x + b_3y + c_3z + d_3 = 0. \end{cases}$$

Assume  $d_1 = A_1x$ ,  $d_2 = A_2x$ ,  $d_3 = A_3x$ . Then

$$\begin{cases} (a_1 + A_1)x + b_1y + c_1z = 0 \\ (a_2 + A_2)x + b_2y + c_2z = 0 \\ (a_3 + A_3)x + b_3y + c_3z = 0 \end{cases}.$$

Eliminate the  $x, y, z$  here visible; then

$$\begin{vmatrix} a_1 + A_1 & b_1 & c_1 \\ a_2 + A_2 & b_2 & c_2 \\ a_3 + A_3 & b_3 & c_3 \end{vmatrix} = 0.$$

The last tablet may be resolved into two, namely:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} A_1 & b_1 & c_1 \\ A_2 & b_2 & c_2 \\ A_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Multiply the former of these by  $x$ , and, as an equivalence, multiply each constituent of the first column of the latter by  $x$ ,

$$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x + \begin{vmatrix} A_1x & b_1 & c_1 \\ A_2x & b_2 & c_2 \\ A_3x & b_3 & c_3 \end{vmatrix} = 0.$$



In the first column of the last restore for  $A_1x$ ,  $A_2x$ ,  $A_3x$  their values  $d_1$ ,  $d_2$ ,  $d_3$ . Then if

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ you have } Dx + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = 0,$$

which solves for  $x$ . By perfectly similar steps

$$Dy + \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_2 & c_3 \end{vmatrix} = 0; \quad Dz + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0.$$

These are easy to remember: each suggests the other, by entire symmetry. The *method* succeeds in Higher Orders.

COR. If we make  $x = \frac{\xi}{u}$ ,  $y = \frac{\eta}{u}$ ,  $z = \frac{\zeta}{u}$ ,

$$\therefore \begin{cases} a_1\xi + b_1\eta + c_1\zeta + d_1u = 0, \\ a_2\xi + b_2\eta + c_2\zeta + d_2u = 0, \\ a_3\xi + b_3\eta + c_3\zeta + d_3u = 0. \end{cases}$$

Then from

$$-Dx \equiv \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}; \quad Dy = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}; \quad -Dz = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

you obtain the proportion

$$= \begin{vmatrix} \xi \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} : - \begin{vmatrix} \eta \\ a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} : \begin{vmatrix} \zeta \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} : - \begin{vmatrix} u \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

15. PROBLEM. To eliminate  $x$  from the two equations

$$\begin{cases} Px^2 + Qx + R = 0 \\ px^2 + qx + r = 0 \end{cases},$$

where  $P$ ,  $Q$ ,  $R$ ,  $p$ ,  $q$ ,  $r$  may involve  $y$  or other quantities. First, put  $x^2 = X$ , then we have

$$\begin{cases} PX + Qx + R = 0 \\ pX + qx + r = 0 \end{cases}.$$

Eliminate  $X$ ; i.e. solve for  $x$ ;

$$\therefore \begin{vmatrix} P & Q \\ p & q \end{vmatrix} x + \begin{vmatrix} P & R \\ p & r \end{vmatrix} = 0 \dots\dots\dots(1).$$

Again, writing in the original

$$\begin{cases} (Px + Q)x + R = 0 \\ (px + q)x + r = 0 \end{cases}$$

eliminate  $x$ , as if  $Px + Q$  and  $px + q$  were ordinary coefficients. Then

$$\begin{vmatrix} Px + Q & R \\ px + q & r \end{vmatrix} = 0,$$

which expanded, gives

$$\begin{vmatrix} Px & R \\ px & r \end{vmatrix} + \begin{vmatrix} Q & R \\ q & r \end{vmatrix} = 0, \text{ or } \begin{vmatrix} P & R \\ p & r \end{vmatrix} x + \begin{vmatrix} Q & R \\ q & r \end{vmatrix} = 0 \dots\dots (2).$$

Eliminate  $x$  between (1) and (2), which gives

$$\begin{vmatrix} \begin{vmatrix} P & Q \\ p & q \end{vmatrix} & \begin{vmatrix} P & R \\ p & r \end{vmatrix} \\ \begin{vmatrix} P & R \\ p & r \end{vmatrix} & \begin{vmatrix} Q & R \\ q & r \end{vmatrix} \end{vmatrix} = 0;$$

that is,

$$\begin{vmatrix} P & Q \\ p & q \end{vmatrix} \cdot \begin{vmatrix} Q & R \\ q & r \end{vmatrix} - \begin{vmatrix} P & R \\ p & r \end{vmatrix}^2 = 0,$$

the result required.

16. PROBLEM. To eliminate  $x$  from the two equations,

$$\begin{cases} ax^3 + 3bx^2 + 3cx + d = 0, \\ ax^2 + 2bx + c = 0. \end{cases}$$

First, multiply the last by  $x$  and subtract;

$$\therefore bx^2 + 2cx + d = 0.$$

Compare the two last with the two equations of Art. 15,

$$\therefore P = a, Q = 2b, R = c, p = b, q = 2c, r = d.$$

Hence the elimination of  $x$  yields

$$\begin{vmatrix} a & 2b \\ b & 2c \end{vmatrix} \cdot \begin{vmatrix} 2b & c \\ 2c & d \end{vmatrix} - \begin{vmatrix} a & c \\ b & d \end{vmatrix}^2 = 0, \text{ or } \begin{vmatrix} a & b \\ b & c \end{vmatrix} \cdot \begin{vmatrix} b & c \\ c & d \end{vmatrix} = \frac{1}{4} \begin{vmatrix} a & c \\ b & d \end{vmatrix}^2.$$

This is the condition of *two equal roots* in the given cubic.

17. By conducting elimination from three given simple equations by a second process, we attain relations which were not easy to anticipate. Assume the three equations of Art. 7. Eliminate  $y$

twice, (1) from the two first; (2) from the second and third. Hence we get

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}; \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} x = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix};$$

from which, by eliminating  $x$ , we obtain  $U=0$ , if we define  $U$  by the equation

$$U = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

But, otherwise eliminated, we find, as the condition of compatibility,  $V_3=0$ .

Therefore  $U=0$  contains the *same relation* of the constituents as  $V_3=0$ . By inspection, we see that  $U$ , equally with  $V_3$ , is linear in  $a_1, b_1, c_1$ . Since then  $U$  and  $V$  vanish together, we have necessarily  $U=\mu \cdot V_3$ , in which  $\mu$  does *not* involve  $a_1, b_1$  nor  $c_1$ . To determine  $\mu$ , suppose  $a_1=1, b_1=0, c_1=0$ ,

$$\therefore U = \begin{vmatrix} a_1 & 0 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

and

$$V_3 = \begin{vmatrix} 1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix};$$

so that in this particular case  $U=b_2 V_3$ , or  $\mu=b_2$ . This then is the value of  $\mu$  for *all* values of  $a_1, b_1, c_1$ , or universally  $U=b_2 \cdot V_3$ , while all the nine constituents are arbitrary. Q.E.D.

18. To remember this important equation, write the square trilinear larger and mark out its minor square. The factor  $b_2$  is in the centre.

$a_1$	$b_1$	$c_1$	for $U = b_2 \cdot V_3$ .
$a_2$	$b_2$	$c_2$	
$a_3$	$b_3$	$c_3$	

By interchanging rows or columns without altering the value of  $V_3$ , fresh relations are obtained. Indeed no one constituent can claim the central place for itself.

*Fourth Order.*

19. To eliminate  $x, y, z, u$  from four given equations each of the form  $a_1x + b_1y + c_1z + d_1u = 0$ , we have now much facility from the Cor. to Art. 14. First, eliminate from the three first equations and get the *proportions* of  $x, y, z, u$ . Next, insert these proportionate values in the fourth equation whereby you entirely eliminate all the four, and obtain an equation  $V_4 = 0$ ; if  $V_4$  stand for

$$a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} - b_4 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + c_4 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - d_4 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This *developed* form of  $V_4$  can always be recovered (by attention to the simple rule given for developing  $V_3$ ) from the conciser or *undeveloped* form

$$V_4 = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Evidently in the definition  $V_4$  is a linear function of  $a_4 b_4 c_4 d_4$ .

So then it must be of any other *row*, the order of the given equations being arbitrary. Also the *first* term of  $V_4$  as defined is free from  $a_1 a_2 a_3$ ; each of the *others* is linear in  $a_1 a_2 a_3$ . Therefore  $V_4$  in square is linear as to its first column; so then must it be as to every column.

Being thus linear, if one column (or row) of such quadrilinear tablet is binomial, it may be split into two tablets by the same process as in the third order. Likewise to multiply any column (or row) by  $m$  multiplies the whole tablet by  $m$ .

To exchange first and second column, exchanges the first and second term of  $V_4$ , but *reverses* their signs. It reverses the sign of the third term, also of the fourth. Thus to exchange the first and second column reverses the sign of  $V_4$ . Evidently then the same must happen by exchanging *any two* contiguous columns. The same argument applies concerning any two contiguous *rows*.

The reasoning of Art. 12 concerning  $V_3$  now applies to  $V_4$ , showing that the tablet vanishes, if one column (or row) be pro-



portional to another column (or row). From this it further follows (as concerning  $V_3$  in Art. 13, and concerning  $V_2$  in Art. 6), that  $V_4$  is not changed in value, if any row (or column) receive increase or decrease in proportion to some other row (or column).

20. That  $V_4$  is not altered by exchanging rows and columns, is generally proved by elaborate inspection of the separate terms when  $V_4$  is resolved into 24 elements each of the form  $a_1 b_2 c_3 d_4$ , no two factors of the same row or column, and showing that the + or - of the term is never altered. It is, no doubt, a perfect demonstration, and more elementary than mine; but I find the less obvious argument of Art. 9 the easier for all the higher orders. Multiply the given equations by  $m_1 n_1 p_1 q_1$  and assign to these multipliers the condition that from the *sum* of the equations thus multiplied  $y, z$  and  $u$  shall disappear.

There will remain  $(ma_1 + na_2 + pa_3 + qa_4)x = 0$ . But our hypothesis forbids  $x = 0$ , hence we have four equations to determine  $mnpq$ , viz.

$$a_1 m + a_2 n + a_3 p + a_4 q = 0,$$

$$b_1 m + b_2 n + b_3 p + b_4 q = 0,$$

$$c_1 m + c_2 n + c_3 p + c_4 q = 0,$$

$$d_1 m + d_2 n + d_3 p + d_4 q = 0.$$

When we eliminate  $mnpq$  the result, which we may call  $S_4 = 0$ , shows  $S_4$  differing from  $V_4$  only in the exchange of rows with columns. Each of them is linear in  $a_1 b_1 c_1 d_1$ . Each involves the same relations between the constituents. The equation  $S_4 = 0$  must be deducible from  $V_4 = 0$ . The only possible relation, making  $S_4$  and  $V_4$  vanish together, has the form  $S_4 = \mu V_4$ , in which  $\mu$  is independent of  $a_1 a_2 a_3 a_4$ . But symmetry proves  $\mu$  equally independent of every other column; therefore  $\mu$  is *numerical*. To find it, we may make the constituents on the positive diagonal all = 1, and all the other constituents vanish. Then both  $S_4$  and  $V_4 = a_1 b_2 c_3 d_4 = 1^5$ . Universally then,  $\mu = 1$ , or  $S_4 = V_4$ . Therefore  $V_4$  is not altered by exchanging rows with columns.

Evidently this argument holds, however high the order of the Tablet, if the successive definitions follow the same law.



22. Take our four equations as in Art. 19. From the three first and also from the three last eliminate both  $y$  and  $z$ ; whence

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = - \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}; \text{ and } \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} x = - \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

To eliminate  $x$  from these two, we have

$$U_4 = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ \dots\dots\dots \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} b_2 & c_2 & d_2 \\ \dots\dots\dots \\ b_4 & c_4 & d_4 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 & c_2 \\ \dots\dots\dots \\ a_4 & b_4 & c_4 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 & d_1 \\ \dots\dots\dots \\ b_3 & c_3 & d_3 \end{vmatrix},$$

which we may remember by

$a_1$	$b_1$	$c_1$	$d_1$
$a_2$	$b_2$	$c_2$	$d_2$
$a_3$	$b_3$	$c_3$	$d_3$
$a_4$	$b_4$	$c_4$	$d_4$

Thus  $U_4 = 0$  and  $V_4 = 0$  express the same condition of the constituents for reconciling the four equations.

Inspection shows that  $U_4$ , like  $V_4$ , is linear in  $a_1 b_1 c_1 d_1$ . Put  $U_4 = \mu V_4$ , and  $\mu$  will be free from these four. Assume then  $a_1 = 1$ ,  $b_1 = c_1 = d_1 = 0$ , and it will not affect  $\mu$ . But it makes

$$V_4 = 1 \cdot \begin{vmatrix} b_2 & c_2 & d_2 \\ \dots\dots\dots \\ b_4 & c_4 & d_4 \end{vmatrix} \text{ and } U_4 = 1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} b_2 & c_2 & d_2 \\ \dots\dots\dots \\ b_4 & c_4 & d_4 \end{vmatrix};$$

that is,

$$\mu = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},$$

whence  $U_4$  generally =  $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \cdot V_4$ . This could not have been foreseen. By varying the order of the elements, we have other results.

Observe that  $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$  is the square in the centre of  $V_4$ . The trilinears in  $U$  are squares cut from the four corners of  $V_4$ ; those of the first term in  $U_4$  being from the positive diagonal.

23. By aid of Art. 21 we readily proceed to the Fifth Order, with the same law of continuous formation; whence in every Order we have these same properties.

(1) To exchange rows and columns does not affect the value of  $V_n$ .

(2)  $V_n$  is a linear function of any one row, or any one column.

(3) If a row or column be binomial, the  $V_n$  may be split into  $V'_n \pm V''_n$ .

(4) To multiply a row or column by  $m$ , multiplies  $V_n$  by  $m$ .

(5) To exchange any row (or column) with a contiguous row (or column) changes  $V_n$  to  $-V_n$ .

(6) If one row (or column) is identical with or proportional to another row (or column), the  $V_n = \text{zero}$ .

(7)  $V_n$  is not altered in value when a row (or column) receives increase or decrease proportioned to another row or column.

(8) If  $V_n$  be divided along the diagonal, so as to fall into four parts, two squares and two rectangular complements, the vanishing of one complement makes the other wholly ineffective.

24. To prove the last universally, it suffices to prove it for the fifth order.

Call the two squares  $P, S$  and the complements  $Q, R$ . Then if one of the complements, as  $Q$ , have all its constituents zero, *I say*,  $R$  is ineffective, and  $V = P \cdot S$ , just as if  $R$  also had all its constituents zero. For every term of  $V_5$  when fully expanded, has the form  $a_m b_n c_p d_q e_r$ , where  $mnpqr$  are taken from 1, 2, 3, 4, 5 and no two are the same. Hence  $a_4 b_4 c_4$  and  $a_5 b_5 c_5$  (the constituents of  $R$ ) are necessarily multiplied by one or other of the zeros ( $d_1 e_1 d_2 e_2 d_3 e_3$ ) in  $Q$ , and



all the products vanish. Consequently, if  $P$  is of the third order and  $S$  of the second, the  $V$  in question is equivalent to

$$V_5 = \begin{vmatrix} a_1 & b_1 & c_1 & \vdots & 0 & 0 \\ a_2 & b_2 & c_2 & \vdots & 0 & 0 \\ a_3 & b_3 & c_3 & \vdots & 0 & 0 \\ \dots\dots\dots & & & & & \\ 0 & 0 & 0 & \vdots & d_4 & e_4 \\ 0 & 0 & 0 & \vdots & d_5 & e_5 \end{vmatrix}$$

and might arise from two equations separately, yielding  $P=0$ , and  $S=0$ . In fact  $V=P.S$ .

If Spottiswoode did not plant the first germ of this very valuable theory, he first investigated the laws and exhibited its vast power.

## TRACT V.

### INTRODUCTION TO TABLES I. AND II.

To these four Elementary Tracts I have added two Numerical Tables, solely because their compilation and verification is elementary.

Table I. gives values of  $A^{-n}$  to 20 decimal places. Here  $A$  means the series 2, 3, 4, ... up to 60, and the odd numbers from 61 to 77; and  $n$  means 1, 2, 3, ... continued until  $A^{-n}$  is about to vanish. To verify, use the formula

$$A^{-1} + A^{-2} + A^{-3} + \dots + A^{-m} + \frac{A^{-m}}{A-1} = \frac{1}{A-1}.$$

The reader may convince himself how searching is this test, by applying it, for instance, to  $A^{-n}$  when  $A=37$  or when  $A=71$ . Only in the case of  $2^{-n}$  and  $3^{-n}$ , where the Tablets give only *odd* values of  $n$ , we must apply the formula

$$(A^{-1} + A^{-3} + A^{-5} + \dots + A^{-2m-1}) = \frac{A - A^{-2m-1}}{A^2 - 1}.$$

Table II. has values of  $x^n$  with 12 decimal places, where  $x$  means .02, .03, .04 up to .50 and  $n$  is continued from 1, 2, 3, ... until  $x^n$  is insignificant. The formula of verification is (with  $m$  any integer less than  $r$ )

$$(x^m + x^{m+1} + x^{m+2} + \dots + x^r) \cdot (1-x) = x^m - x^{r+1}.$$

I compiled this table while working at Spence's integral

$$\int_0^1 \log(1+x) \frac{dx}{x},$$

but it has much wider use.

One who is sagely incredulous of printed tables can verify *for himself* any tablet which he is disposed to use with much greater ease than he could compose the tablet. Thus, too, he would detect any error from miscopying or misprinting, against which I can least give a guarantee.

$n$	$2^{-n}$ ( $n$ odd).	$n$	$3^{-n}$ ( $n$ odd).
1	.5	1	.33333 33333 33333 33333
3	.125	3	.03703 70370 37037 03703
5	.03125	5	.411 52263 37448 55967
7	.00781 25	7	.45 72473 70827 61774
9	.00195 3125	9	.5 08052 63425 29086
11	.48 82812 5	11	..... 56450 29269 47676
13	.12 20703 125	13	..... 6272 25474 38630
15	.3 05175 78125	15	..... 696 91719 37625
17	..... 76293 94531 25	17	..... 77 43524 37514
19	..... 19073 48632 8125	19	..... 8 60391 59724
21	..... 4768 37158 20312	21	..... 95599 06636
23	..... 1192 09289 55078	23	..... 10622 11848
25	..... 298 02322 38769	25	..... 1180 23539
27	..... 74 50580 59692	27	..... 131 13726
29	..... 18 62645 14923	29	..... 14 57081

$n$	$4^{-n}$	$n$	$4^{-n}$
1	.25	17	.00000 00000 58207 66091
2	.0625	18	..... 14551 91523
3	.01562 5	19	..... 3637 97880
4	.00390 625	20	..... 909 49470
5	.97 65625	21	..... 227 37367
6	.24 41406 25	22	..... 56 84342
7	.6 10351 5625	23	..... 14 21085
8	.1 52587 89062 5	24	..... 3 55271
9	..... 38146 97265 625	25	..... 88818
10	..... 9536 74316 40625	26	..... 22204
11	..... 2384 18579 10156	27	..... 5551
12	..... 596 04644 77539	28	..... 1388
13	..... 149 01161 10385	29	..... 347
14	..... 37 25290 29846	30	..... 87
15	..... 9 31322 57461	31	..... 22
16	..... 2 32830 64365	32	..... 5
		33	..... 1

TABLE I. TWENTY DECIMALS.

$n$	$5^{-n}$	$n$	$6^{-n}$
1	.2	1	.16666 66666 66666 66666
2	.04	2	.02777 77777 77777 77777
3	.008	3	462 96296 29629 62962
4	.00160	4	77 16049 38271 60494
5	.00032	5	12 86008 23045 26749
6	.00006 4	6	2 14334 70507 54458
7	.00001 28	7	..... 35722 45084 59076
8	..... 256	8	..... 5933 74180 76513
9	..... 05120	9	..... 992 29030 12752
10	..... 01024	10	..... 165 38171 68792
11	..... 00204 8	11	..... 27 53661 97499
12	..... 40 96	12	..... 4 59393 65799
13	..... 8 192	13	..... ..... 76565 60966
14	..... 1 63840	14	..... ..... 12760 93494
15	..... ..... 32768	15	..... ..... 2126 82249
16	..... ..... 6553 6	16	..... .. 354 47041
17	..... ..... 1310 72	17	..... ..... 59 07840
18	..... ..... 262 144	18	..... ..... 9 84640
19	..... ..... 52 42880	19	..... ..... 1 64106
20	..... ..... 10 48576	20	..... ..... ..... 27351
21	..... ..... 2 09715	21	..... ..... ..... 4558
22	..... ..... ..... 41943	22	..... ..... ..... 759
23	..... ..... ..... 8388	23	..... ..... ..... 126
24	..... ..... ..... 1677	24	..... ..... ..... 21
25	..... ..... ..... 335	25	..... ..... ..... 3
26	..... ..... ..... 67		
27	..... ..... ..... 13		
28	..... ..... ..... 2		

$n$	$7^{-n}$	$n$	$7^{-n}$
1	.14285 71428 57142 85714	12	.00000 00000 72247 61581
2	.02040 81632 65306 12245	13	..... ..... 10321 08797
3	291 54518 95043 73178	14	..... ..... 1474 44114
4	41 64931 27863 39025	15	..... ..... 210 63445
5	5 94990 18266 19861	16	..... ..... 30 09063
6	..... 84998 59752 31409	17	..... ..... 4 29866
7	..... 12142 65678 90201	18	..... ..... ..... 61409
8	..... 1734 66525 55743	19	..... ..... ..... 8774
9	..... 247 80932 22249	20	..... ..... ..... 1253
10	..... 35 40133 17464	21	..... ..... ..... 179
11	..... 5 05733 31066	22	..... ..... ..... 25
		23	..... ..... ..... 4



TABLE I. TWENTY DECIMALS.

$n$	$8^{-n}$				$n$	$9^{-n}$			
1	.125				1	.11111 11111 11111 11111			
2	.01562 5				2	.01234 56790 12345 67901			
3	.00195 3125				3	.0137 17421 12482 85322			
4	.00024 41406 25				4	.015 24157 90275 87258			
5	.00003 05175 78125				5	.01 69350 87808 43029			
6	.00000 38146 97265 625				6	.00000 18816 76423 15892			
7	.00000 4768 37158 20312				7	.00000 2090 75158 12877			
8	.00000 596 04644 77539				8	.00000 232 30573 12542			
9	.00000 74 50580 59692				9	.00000 25 81174 79171			
10	.00000 9 31322 57461				10	.00000 2 86797 19908			
11	.00000 1 16415 32182				11	.00000 31866 35545			
12	.00000 14551 91523				12	.00000 3540 70616			
13	.00000 1818 98940				13	.00000 393 41179			
14	.00000 227 37367				14	.00000 43 71242			
15	.00000 28 42171				15	.00000 4 85693			
16	.00000 3 55271				16	.00000 53966			
17	.00000 44409				17	.00000 5996			
18	.00000 5551				18	.00000 666			
19	.00000 694				19	.00000 74			
20	.00000 87				20	.00000 8			
21	.00000 11				21	.00000 0.9			
22	.00000 1								

$n$	$11^{-n}$				$n$	$12^{-n}$			
1	.09090 90909 09090 90909				1	.08333 33333 33333 33333			
2	.00826 44628 09917 35537				2	.00694 44444 44444 44444			
3	.00075 13148 00901 57776				3	.00057 87037 03703 70370			
4	.00006 83013 45536 50707				4	.00004 82253 08641 97531			
5	.00000 62092 13230 59153				5	.00000 40187 75720 16461			
6	.00000 5644 73930 05377				6	.00000 3348 97976 68038			
7	.00000 513 15811 82307				7	.00000 279 08164 72336			
8	.00000 46 65073 80209				8	.00000 23 25680 39361			
9	.00000 4 24097 61837				9	.00000 1 93806 69946			
10	.00000 38554 32894				10	.00000 16150 55829			
11	.00000 3504 93899				11	.00000 1345 87985			
12	.00000 318 63082				12	.00000 112 15665			
13	.00000 28 96644				13	.00000 9 34639			
14	.00000 2 63331				14	.00000 77886			
15	.00000 23939				15	.00000 6490			
16	.00000 2176				16	.00000 541			
17	.00000 198				17	.00000 45			
18	.00000 18				18	.00000 4			
19	.00000 1								

$n$	$13^{-n}$				$n$	$14^{-n}$			
1	07692	30769	23076	92307	1	07142	85714	28571	42857
2	591	71597	63313	60947	2	510	20408	16326	53061
3	45	51661	35639	50842	3	36	44314	86880	46647
4	3	50127	79664	57757	4	2	60308	20491	46189
5	.....	26932	90743	42904	5	.....	18593	44320	81870
6	.....	2071	76211	03300	6	.....	1328	10308	62991
7	.....	159	36631	61792	7	.....	94	86450	61642
8	.....	12	25894	73984	8	.....	6	77603	61546
9	.....	.....	94299	59537	9	.....	.....	48400	25825
10	.....	.....	7253	81503	10	.....	.....	3457	16130
11	.....	.....	557	98577	11	.....	.....	246	94009
12	.....	.....	42	92198	12	.....	.....	17	63858
13	.....	.....	3	30169	13	.....	.....	1	25989
14	.....	.....	.....	25397	14	.....	.....	.....	8999
15	.....	.....	.....	1594	15	.....	.....	.....	643
16	.....	.....	.....	150	16	.....	.....	.....	46
17	.....	.....	.....	11	17	.....	.....	.....	3

$n$	$15^{-n}$				$n$	$17^{-n}$			
1	06666	66666	66666	66666	1	05882	35294	11764	70588
2	444	44444	44444	44444	2	346	02076	12456	74740
3	29	62962	96296	29629	3	20	35416	24262	16161
4	1	97530	86419	75308	4	1	19730	36721	30362
5	.....	13168	72427	98360	5	.....	7042	96277	72375
6	.....	877	91495	19891	6	.....	414	29192	80728
7	.....	58	52766	34659	7	.....	24	37011	34160
8	.....	3	90184	42311	8	.....	1	43353	60833
9	.....	.....	26012	29487	9	.....	.....	8432	56519
10	.....	.....	1734	15299	10	.....	.....	496	03324
11	.....	.....	115	61020	11	.....	.....	29	17842
12	.....	.....	7	70734	12	.....	.....	1	71638
13	.....	.....	.....	51382	13	.....	.....	.....	10096
14	.....	.....	.....	3425	14	.....	.....	.....	594
15	.....	.....	.....	228	15	.....	.....	.....	35
16	.....	.....	.....	15	16	.....	.....	.....	2
17	.....	.....	.....	1					

For  $16^{-n}$  look to  $4^{-2n}$ .

$n$	$18^{-n}$				$n$	$19^{-n}$			
1	05555	55555	55555	55555	1	05263	15789	47368	42105
2	308	64197	53086	41975	2	277	00831	02493	07479
3	17	14677	64060	35665	3	14	57938	47499	63551
4	.....	95259	86892	24204	4	.....	76733	60394	71766
5	.....	5292	21494	01345	5	.....	4038	61073	40619
6	.....	294	01194	11186	6	.....	212	55845	96874
7	.....	16	33399	67288	7	.....	11	18728	73519
8	.....	.....	90744	42627	8	.....	.....	58880	45975
9	.....	.....	5041	35701	9	.....	.....	3098	97156
10	.....	.....	280	07539	10	.....	.....	163	10376
11	.....	.....	15	55974	11	.....	.....	8	58441
12	.....	.....	.....	86443	12	.....	.....	.....	45181
13	.....	.....	.....	4802	13	.....	.....	.....	2378
14	.....	.....	.....	267	14	.....	.....	.....	125
15	.....	.....	.....	15	15	.....	.....	.....	6

$n$	$21^{-n}$				$n$	$22^{-n}$			
1	04761	90476	19047	61904	1	04545	45454	54545	45454
2	226	75736	96145	12471	2	206	61157	02479	33884
3	10	79796	99816	43451	3	9	39143	50112	69722
4	.....	51418	90467	44926	4	.....	42688	34096	03169
5	.....	2448	51927	02139	5	.....	1940	37913	45599
6	.....	116	59615	57245	6	.....	88	19905	15709
7	.....	5	55219	78916	7	.....	4	00904	77987
8	.....	.....	26439	03758	8	.....	.....	18222	94454
9	.....	.....	1259	00179	9	.....	.....	828	31566
10	.....	.....	59	95246	10	.....	.....	37	65071
11	.....	.....	2	85488	11	.....	.....	1	71139
12	.....	.....	.....	13594	12	.....	.....	.....	7779
13	.....	.....	.....	647	13	.....	.....	.....	353
14	.....	.....	.....	31	14	.....	.....	.....	16
15	.....	.....	.....	1					

TABLE I. TWENTY DECIMALS.

$n$	$23^{-n}$				$n$	$24^{-n}$			
1	04347	82608	69565	21739	1	04166	66666	66666	66666
2	189	03591	68241	96597	2	173	61111	11111	11111
3	8	21895	29053	99852	3	7	23379	62962	96296
4	.....	35734	57784	95646	4	.....	30140	81790	12345
5	.....	1553	67729	78071	5	.....	1255	86741	25514
6	.....	67	55118	68612	6	.....	52	32780	88563
7	.....	2	93700	81244	7	.....	2	18032	53690
8	.....	.....	12769	60054	8	.....	.....	9084	68903
9	.....	.....	555	20002	9	.....	.....	378	52871
10	.....	.....	24	13913	10	.....	.....	15	77203
11	.....	.....	1	04953	11	.....	.....	.....	65716
12	.....	.....	.....	4563	12	.....	.....	.....	2738
13	.....	.....	.....	198	13	.....	.....	.....	114
14	.....	.....	.....	8	14	.....	.....	.....	4

For  $25^{-n}$  look to  $5^{-2n}$ .

$n$	$26^{-n}$				$n$	$27^{-n}$			
1	03846	15384	61538	46153	1	03703	70370	37037	03703
2	147	92899	40828	40236	2	137	17421	12482	85322
3	5	68957	66954	93855	3	5	08052	63425	29086
4	.....	21882	98729	03609	4	.....	18816	76423	15892
5	.....	841	65335	73216	5	.....	696	91719	37625
6	.....	32	37128	29739	6	.....	25	81174	79171
7	.....	1	24504	93451	7	.....	.....	95599	06636
8	.....	.....	4788	65133	8	.....	.....	3540	70616
9	.....	.....	184	17890	9	.....	.....	131	13726
10	.....	.....	7	08380	10	.....	.....	4	85693
11	.....	.....	.....	27245	11	.....	.....	.....	17982
12	.....	.....	.....	1048	12	.....	.....	.....	666
13	.....	.....	.....	40	13	.....	.....	.....	25
14	.....	.....	.....	1					



$n$	$28^{-n}$	$n$	$29^{-n}$
1	·03571 42857 14285 71428	1	·03448 27586 20689 65517
2	127 55102 04081 63265	2	118 90606 42092 74673
3	4 55539 35860 05831	3	4 10020 91106 64644
4	..... 16269 26280 71637	4	..... 14138 65210 57401
5	..... 581 04510 02558	5	..... 487 53972 77841
6	..... 20 75161 07234	6	..... 16 81171 47512
7	..... 74112 89544	7	..... 57971 43011
8	..... 2646 88912	8	..... 1999 01483
9	..... 94 53175	9	..... 68 93154
10	..... 3 37613	10	..... 2 37695
11	..... 12057	11	..... 8196
12	..... 430	12	..... 283
13	..... 15	13	..... 9

$n$	$31^{-n}$	$n$	$32^{-n}$
1	·03225 80645 16129 03226	1	·03125
2	104 05827 26326 74298	2	97 65625
3	3 35671 84720 21751	3	3 05175 78125
4	..... 10828 12410 32960	4	..... 9536 74316 40625
5	..... 349 29432 59127	5	..... 298 02322 38769
6	..... 11 26755 89004	6	..... 9 31322 57461
7	..... 36346 96419	7	..... 29103 83045
8	..... 1172 48271	8	..... 909 49470
9	..... 37 82202	9	..... 28 42171
10	..... 1 22007	10	..... 88818
11	..... 3936	11	..... 2775
12	..... 127	12	..... 86
13	..... 4	13	..... 2

$n$	$33^{-n}$				$n$	$34^{-n}$			
1	03030	30303	03030	30303	1	02941	17647	05882	35294
2	91	82736	45546	37282	2	86	50519	03114	18685
3	2	78264	74107	46584	3	2	54427	03032	77020
4	.....	8432	26488	10502	4	.....	7483	14795	08147
5	.....	255	52317	82136	5	.....	220	09258	67888
6	.....	7	74312	66125	6	.....	6	47331	13761
7	.....	.....	23464	02004	7	.....	.....	19039	15110
8	.....	.....	711	03091	8	.....	.....	559	97503
9	.....	.....	21	54639	9	.....	.....	16	46985
10	.....	.....	.....	65292	10	.....	.....	.....	48441
11	.....	.....	.....	1978	11	.....	.....	.....	1425
12	.....	.....	.....	59	12	.....	.....	.....	42
13	.....	.....	.....	2	13	.....	.....	.....	1

$n$	$35^{-n}$				$n$	$37^{-n}$			
1	02857	14285	71428	57142	1	02702	70270	27027	02702
2	81	63265	30612	24489	2	73	04601	89919	64938
3	2	33236	15160	34985	3	1	97421	67295	12566
4	.....	6663	89004	58142	4	.....	5335	72089	05745
5	.....	190	39685	84518	5	.....	144	20867	27182
6	.....	5	43991	02415	6	.....	3	89753	16951
7	.....	.....	15542	60069	7	.....	.....	10533	86945
8	.....	.....	444	07430	8	.....	.....	284	69917
9	.....	.....	12	68783	9	.....	.....	7	69457
10	.....	.....	.....	36251	10	.....	.....	.....	20796
11	.....	.....	.....	1036	11	.....	.....	.....	562
12	.....	.....	.....	29	12	.....	.....	.....	15
13	.....	.....	.....	0,8					

For  $36^{-n}$  look to  $6^{-2n}$ .

TABLE I. TWENTY DECIMALS.

$n$	$38^{-n}$				$n$	$39^{-n}$			
1	02631	57894	73684	21052	1	02564	10256	41025	64102
2	69	25207	75623	26870	2	65	74621	05923	73438
3	1	82242	30937	45444	3	1	68580	05023	68549
4	.....	4795	85024	66985	4	.....	4322	56539	06886
5	.....	126	20658	54394	5	.....	110	83501	00176
6	.....	3	32122	59326	6	.....	2	84192	33338
7	.....	.....	8740	06824	7	.....	.....	7286	98290
8	.....	.....	230	00179	8	.....	.....	186	84571
9	.....	.....	6	05268	9	.....	.....	4	79091
10	.....	.....	.....	15928	10	.....	.....	.....	12284
11	.....	.....	.....	419	11	.....	.....	.....	315
12	.....	.....	.....	11	12	.....	.....	.....	8

$n$	$41^{-n}$				$n$	$42^{-n}$			
1	02439	02439	02439	02439	1	02380	95238	09523	80952
2	59	48839	97620	46401	2	56	68934	24036	28118
3	1	45093	65795	62107	3	1	34974	62477	05431
4	.....	3538	86970	62490	4	... ..	3213	68154	21558
5	.....	86	31389	52744	5	.....	76	51622	71942
6	.....	2	10521	69579	6	.....	1	82181	49332
7	.....	.....	5134	67550	7	.....	.....	4337	65460
8	.....	.....	125	23599	8	.....	.....	103	27749
9	.....	.....	3	05453	9	.....	.....	2	45899
10	.....	.....	.....	7450	10	.....	.....	.....	5854
11	.....	.....	.....	182	11	.....	.....	.....	139
12	.....	.....	.....	4	12	.....	.....	.....	3

TABLE I. TWENTY DECIMALS.

$n$	$43^{-n}$	$n$	$44^{-n}$
1	·02325 58139 53488 37209	1	·02272 72727 27272 72727
2	54 08328 82639 26447	2	51 65289 25619 83471
3	I 25775 08898 58754	3	I 17392 93764 08715
4	..... 2925 00206 94389	4	..... 2668 02131 00198
5	..... 68 02330 39404	5	..... 60 63684 79549
6	..... I 58193 73009	6	..... I 37811 01808
7	..... ..... 3678 92395	7	..... ..... 3132 06859
8	..... ..... 85 55637	8	..... ..... 71 18337
9	..... ..... I 98968	9	..... ..... I 61780
10	..... ..... ..... 4627	10	..... ..... ..... 3677
11	..... ..... ..... 107	11	..... ..... ..... 84
12	..... ..... ..... 2	12	..... ..... ..... 2

$n$	$45^{-n}$	$n$	$46^{-n}$
1	·02222 22222 22222 22222	1	·02173 91304 34782 60869
2	49 38271 60493 82716	2	47 25897 92060 49149
3	I 09739 36899 86282	3	I 02736 91131 74981
4	..... 2438 65264 44139	4	..... 2233 41111 55977
5	..... 54 19228 09869	5	..... 48 55241 55565
6	..... I 20427 29108	6	..... I 05548 72947
7	..... ..... 2676 16202	7	..... ..... 2294 53759
8	..... ..... 59 47026	8	..... ..... 49 88125
9	..... ..... I 32156	9	..... ..... I 08438
10	..... ..... ..... 2937	10	..... ..... ..... 2357
11	..... ..... ..... 65	11	..... ..... ..... 51
12	..... ..... ..... I	12	..... ..... ..... I



$n$	$47^{-n}$	$n$	$48^{-n}$
1	·02127 65957 44680 85106	1	·02083 33333 33333 33333
2	45 26935 26482 57130	2	43 40277 77777 77777
3	..... 96317 77159 20364	3	..... 90422 45370 37037
4	..... 2049 31428 91923	4	..... 1883 80111 88271
5	..... 43 60243 16849	5	..... 39 24585 66422
6	..... ..... 92771 13124	6	..... ..... 81762 20136
7	..... ..... 1973 85385	7	..... ..... 1703 37919
8	..... ..... 41 99689	8	..... ..... 35 48706
9	..... ..... ..... 89355	9	..... ..... ..... 73931
10	..... ..... ..... 1901	10	..... ..... ..... 1540
11	..... ..... ..... 40	11	..... ..... ..... 32

For  $49^{-n}$  look to  $7^{-2n}$ .

$n$	$51^{-n}$	$n$	$52^{-n}$
1	·01960 78431 37254 90196	1	·01923 07692 30769 23076
2	38 44675 12495 19415	2	36 98224 85207 10059
3	..... 75385 78676 37636	3	..... 71119 70869 36732
4	..... 1478 15268 16424	4	..... 1367 68670 56475
5	..... 28 98338 59143	5	..... 26 30166 74163
6	..... ..... 56830 16846	6	..... ..... 50580 12965
7	..... ..... 1114 31703	7	..... ..... 972 69480
8	..... ..... 21 84935	8	..... ..... 18 70567
9	..... ..... ..... 42842	9	..... ..... ..... 35972
10	..... ..... ..... 840	10	..... ..... ..... 692
11	..... ..... ..... 16	11	..... ..... ..... 13

TABLE I. TWENTY DECIMALS.

$n$	$53^{-n}$					$n$	$54^{-n}$				
1	01886	79245	28301	88679		1	01851	85185	18518	51852	
2	35	59985	76005	69598		2	34	29355	28120	71331	
3	.....	67169	54264	25841		3	.....	63506	57928	16136	
4	.....	1267	34986	11808		4	.....	1176	04776	44743	
5	.....	23	91226	15317		5	.. ..	21	77866	23051	
6	.....	.....	45117	47459		6	.....	.....	40330	85612	
7	.....	.....	851	27310		7	.....	.....	746	86769	
8	.....	.....	16	06176		8	.....	.....	13	83088	
9	.....	.....	.....	30305		9	.....	.....	.....	25613	
10	.....	.....	.....	572		10	.....	.....	.....	474	
11	.....	.....	.....	11		11	.....	.....	.....	9	

$n$	$55^{-n}$					$n$	$56^{-n}$				
1	01818	18181	81818	18181		1	01785	71428	57142	85714	
2	33	05785	12396	69421		2	31	88775	51020	40816	
3	.....	60105	18407	21262		3	.....	56942	41982	50729	
4	.....	1092	82152	85841		4	.....	1016	82892	54477	
5	.....	19	86948	23379		5	.....	18	15765	93829	
6	.....	.....	36126	33152		6	.....	.....	32424	39175	
7	.....	.....	656	84239		7	.....	.....	579	00699	
8	.....	.....	11	94259		8	.....	.....	10	33941	
9	.....	.....	.....	21714		9	.....	.....	.....	18463	
10	.....	.....	.....	386		10	.....	.....	.....	330	
11	.....	.....	.....	7		11	.....	.....	.....	6	

$n$	$57^{-n}$					$n$	$58^{-n}$				
1	01754	38596	49122	80701		1	01724	13793	10344	82758	
2	30	77870	11388	11942		2	29	72651	60523	18668	
3	.....	53997	72129	61613		3	.....	51252	61388	33082	
4	.....	947	32844	37923		4	.....	883	66575	66087	
5	.....	16	61979	72595		5	.....	15	23561	64932	
6	.....	.....	29157	53905		6	.....	.....	26268	30429	
7	.....	.....	511	53577		7	.....	.....	452	90179	
8	.....	.....	8	97431		8	.....	.....	7	80865	
9	.....	.....	.....	15744		9	.....	.....	.....	13463	
10	.....	.....	.....	276		10	.....	.....	.....	232	
11	.....	.....	.....	5		11	.....	.....	.....	4	

TABLE I. TWENTY DECIMALS.

$n$	$59^{-n}$				$n$	$61^{-n}$			
1	01694	91525	42372	88136	1	01639	34426	22950	81967
2	28	72737	71904	62511	2	26	87449	61031	98065
3	.....	48690	46981	43432	3	.....	44056	55098	88493
4	.....	825	26220	02431	4	.....	722	23854	08008
5	.....	13	98749	49194	5	.....	11	83997	60787
6	.....	.....	23707	61851	6	.....	.....	19409	79685
7	.....	.....	401	82404	7	.....	.....	318	19339
8	.....	.....	6	81058	8	.....	.....	5	21628
9	.....	.....	.....	11543	9	.....	.....	.....	8551
10	.....	.....	.....	196	10	.....	.....	.....	140
11	.....	.....	.....	3	11	.....	.....	.....	2

$n$	$63^{-n}$				$n$	$65^{-n}$			
1	01587	30158	73015	87301	1	01538	46153	84615	38461
2	25	19526	32905	01385	2	23	66863	90532	54438
3	.....	39992	48141	34942	3	.....	36413	29085	11607
4	.....	634	80129	22777	4	.....	560	20447	46333
5	.....	10	07621	09885	5	.....	8	61853	03789
6	.....	.....	15993	98569	6	.....	.....	13259	27751
7	.....	.....	253	87279	7	.....	.....	203	98888
8	.....	.....	4	02973	8	.....	.....	3	13829
9	.....	.....	.....	6396	9	.....	.....	.....	4828
10	.....	.....	.....	101	10	.....	.....	.....	75

$n$	$67^{-n}$				$n$	$69^{-n}$			
1	01492	53731	34328	35821	1	01449	27536	23188	40579
2	22	27667	63198	93072	2	21	00399	07582	44066
3	.....	33248	77062	67061	3	.....	30040	56631	62957
4	.....	496	25030	78613	4	.....	441	16762	77724
5	.....	7	40672	10128	5	.....	6	39373	37358
6	.....	.....	11054	80748	6	.....	.....	9266	28077
7	.....	.....	164	99712	7	.....	.....	134	29392
8	.....	.....	2	46264	8	.....	.....	1	94629
9	.....	.....	.....	3675	9	.....	.....	.....	2821
10	.....	.....	.....	55	10	.....	.....	.....	40

$n$	$71^{-n}$				$n$	$73^{-n}$			
1	01408	45070	42253	52213	1	01369	86301	36986	30137
2	19	83733	38623	28903	2	18	76524	67629	94933
3	.....	27939	90684	83506	3	.....	25705	81748	35547
4	.....	393	51981	47655	4	.....	352	13448	60761
5	.....	5	54253	26023	5	.....	4	82376	00832
6	.....	.....	7806	38395	6	.....	.....	6607	89052
7	.....	.....	109	94907	7	.....	.....	90	51905
8	.....	.....	1	54858	8	.....	.....	1	23999
9	.....	.....	.....	2181	9	.....	.....	.....	1699
10	.....	.....	.....	31	10	.....	.....	.....	23

$n$	$75^{-n}$				$n$	$77^{-n}$			
1	01333	33333	33333	33333	1	01298	70129	87012	98701
2	17	77777	77777	77777	2	16	86625	06324	84384
3	.....	23703	70370	37037	3	.....	21904	22160	06291
4	.....	316	04938	27160	4	.....	284	47041	03977
5	.....	4	21399	17695	5	.....	3	69442	09142
6	.....	.....	5618	65569	6	.....	.....	4797	94924
7	.....	.....	74	91541	7	.....	.....	62	31103
8	.....	.....	.....	99887	8	.....	.....	.....	80923
9	.....	.....	.....	1339	9	.....	.....	.....	1051
10	.....	.....	.....	18	10	.....	.....	.....	14



TABLE II. Powers of  $\cdot 02$ ,  $\cdot 03$ ,  $\cdot 04$ , ... up to  $\cdot 50$ , useful to compute  $A_1x + A_2x^2 + A_3x^3 + \&c.$ , when  $x$  does not exceed  $\frac{1}{2}$ .

(Twelve Decimals.)

$n$	$(\cdot 02)^n$	$(\cdot 03)^n$	$(\cdot 04)^n$	$n$
2	$\cdot 0004$	$\cdot 0009$	$\cdot 0016$	2
3	$\cdot 0\ 00\ 08$	..... 27	..... 64	3
4	..... 0016	..... 0081	..... 0256	4
5	..... 32	..... 2 43	..... 10 24	5
6	..... 0064	..... 0729	..... 4096	6
7	..... 01	..... 0022	..... 164	7
8			..... 6	8

$n$	$(\cdot 05)^n$	$(\cdot 06)^n$	$(\cdot 07)^n$	$n$
2	$\cdot 0025$	$\cdot 0036$	$\cdot 0049$	2
3	1 25	2 16	3 43	3
4	..... 0625	..... 1296	..... 2401	4
5	..... 31 25	..... 77 76	..... 168 07	5
6	..... 1 5625	..... 4 6656	..... 11 7649	6
7	..... 781	..... 2799	..... 8235	7
8	..... 39	..... 168	..... 576	8
9	..... 2	..... 10	..... 40	9
10			..... 3	10

$n$	$(\cdot 08)^n$	$(\cdot 09)^n$	$(\cdot 11)^n$	$n$
2	$\cdot 0064$	$\cdot 0081$	$\cdot 0121$	2
3	5 12	7 29	13 31	3
4	..... 4096	..... 6561	1 4641	4
5	..... 327 05	..... 590 49	..... 1610 51	5
6	..... 26 2144	..... 53 1441	..... 177 1561	6
7	..... 2 0971	..... 4 7830	..... 19 4871	7
8	..... 1678	..... 4305	..... 2 1436	8
9	..... 134	..... 387	..... 2358	9
10	..... 11	..... 35	..... 259	10
11	..... 1	..... 3	..... 28	11
12			..... 3	12

TABLE II. TWELVE DECIMALS.

$n$	$(\cdot 12)^n$	$(\cdot 13)^n$	$(\cdot 14)^n$	$n$
2	0144	0169	0196	2
3	17 28	21 97	27 44	3
4	2 0736	2 8561	3 8416	4
5	..... 2488 32	..... 3172 93	..... 5378 24	5
6	..... 298 5984	..... 482 6809	..... 752 9536	6
7	..... 35 8318	..... 62 7485	..... 105 4135	7
8	..... 4 2988	..... 8 1573	..... 14 7579	8
9	..... 5160	..... 1 0604	..... 2 0661	9
10	..... 619	..... 1378	..... 2892	10
11	..... 74	..... 179	..... 405	11
12	..... 8	..... 23	..... 57	12
13	..... 1	..... 3	..... 8	13
14			..... 1	14

$n$	$(\cdot 15)^n$	$(\cdot 16)^n$	$(\cdot 17)^n$	$n$
2	0225	0256	0289	2
3	38 75	40 96	49 13	3
4	5 0625	6 5538	8 3521	4
5	..... 7593 75	1 0485 76	1 4198 57	5
6	..... 1139 0625	..... 1677 7216	..... 2413 7569	6
7	..... 170 8594	..... 268 4354	..... 410 3387	7
8	..... 25 6289	..... 42 9497	..... 69 7576	8
9	..... 3 8443	..... 6 8719	..... 11 8588	9
10	..... 5766	..... 1 0995	..... 2 0160	10
11	..... 864	..... 1759	..... 3427	11
12	..... 130	..... 281	..... 583	12
13	..... 19	..... 45	..... 99	13
14	..... 3	..... 7	..... 17	14
15		..... 1	..... 3	15

TABLE II. TWELVE DECIMALS.

$n$	$(\cdot 18)^n$	$(\cdot 19)^n$	$(\cdot 20)^n$	$n$
2	$\cdot 0324$	$\cdot 0361$	$\cdot 04$	2
3	58 32	68 59	$\cdot 008$	3
4	10 4976	13 0321	$\cdot 0016$	4
5	1 8895 68	2 4760 99	$\cdot 0003$ 2	5
6	..... 3401 2224	..... 4704 5881	..... 64	6
7	..... 612 2200	..... 893 8717	..... 128	7
8	..... 110 1996	..... 169 8356	..... 0256	8
9	..... 19 8359	..... 32 2688	..... 0051 2	9
10	..... 3 5705	..... 6 1311	..... 0010 24	10
11	..... 6427	..... 1 1649	..... 0002 048	11
12	..... 1157	..... 2213	..... 4096	12
13	..... 208	..... 421	..... 819	13
14	..... 37	..... 80	..... 164	14
15	..... 7	..... 15	..... 33	15
16	..... 1	..... 3	..... 1	16

$n$	$(\cdot 21)^n$	$(\cdot 22)^n$	$(\cdot 23)^n$	$n$
2	$\cdot 0441$	$\cdot 0484$	$\cdot 0529$	2
3	92 61	$\cdot 0108$ 48	$\cdot 0121$ 67	3
4	19 4481	23 4236	27 9841	4
5	4 0841 01	5 1536 32	6 4363 43	5
6	..... 8576 6121	1 1337 9904	1 4803 5889	6
7	..... 1801 0885	..... 2494 3579	..... 3404 8254	7
8	..... 378 2286	..... 548 7587	..... 783 1098	8
9	..... 79 4280	..... 79 4280	..... 180 1153	9
10	..... 16 6799	..... 26 5599	..... 41 4265	10
11	..... 3 5028	..... 5 8432	..... 9 5281	11
12	..... 7356	..... 1 2855	..... 2 1914	12
13	..... 1545	..... 2828	..... 5040	13
14	..... 324	..... 622	..... 1159	14
15	..... 68	..... 137	..... 267	15
16	..... 14	..... 30	..... 61	16
17	..... 3	..... 6,6	..... 14	17
18		..... 1,6	..... 3	18

TABLE II. TWELVE DECIMALS.

$n$	$(\cdot 24)^n$	$(\cdot 25)^n$	$(\cdot 26)^n$	$n$
2	0576	0625	0676	2
3	0138 24	0156 25	0175 76	3
4	33 1776	0039 0625	0045 6976	4
5	7 9626 24	9 7656 25	0011 8813 76	5
6	1 9110 2976	2 4414 0625	3 0891 5776	6
7	..... 4586 4714	..... 6103 5156	..... 8031 8102	7
8	..... 1100 7531	..... 1525 8789	..... 2088 2706	8
9	..... 264 1807	..... 381 4697	..... 542 9504	9
10	..... 63 4034	..... 95 3674	..... 141 1671	10
11	..... 15 2168	..... 23 8418	..... 36 7034	11
12	..... 3 6520	..... 5 9604	..... 9 5429	12
13	..... 8765	..... 1 4901	..... 2 4811	13
14	..... 2103	..... 3725	..... 6451	14
15	..... 505	..... 931	..... 1677	15
16	..... 121	..... 233	..... 436	16
17	..... 29	..... 58	..... 113	17
18	..... 7	..... 14	..... 29	18
19	..... 1,7	..... 4	..... 7	19
20		..... 1	..... 2	20

$n$	$(\cdot 27)^n$	$(\cdot 28)^n$	$(\cdot 29)^n$	$n$
2	0729	0784	0841	2
3	0196 83	0219 52	0243 89	3
4	0053 1441	0061 4656	0070 7281	4
5	0014 3489 07	0017 2103 68	0020 5111 49	5
6	3 8742 0489	4 8189 0304	5 9482 3321	6
7	1 0460 3532	1 3492 9285	1 7249 8763	7
8	..... 2824 2954	..... 3778 0200	..... 5002 4641	8
9	..... 762 5597	..... 1057 8456	..... 1450 7146	9
10	..... 205 8911	..... 296 1968	..... 420 7072	10
11	..... 55 5906	..... 82 9351	..... 122 0051	11
12	..... 15 0095	..... 23 2218	..... 35 3817	12
13	..... 4 0525	..... 6 5021	..... 10 2607	13
14	..... 1 0942	..... 1 8026	..... 2 9756	14
15	..... 2954	..... 5098	..... 8629	15
16	..... 797	..... 1427	..... 2502	16
17	..... 215	..... 400	..... 726	17
18	..... 58	..... 112	..... 210	18
19	..... 16	..... 31	..... 61	19
20	..... 5	..... 9	..... 18	20
21	..... 1	..... 2	..... 5	21
22			..... 1	22



$n$	$(\cdot 30)^n$	$(\cdot 31)^n$	$(\cdot 32)^n$	$n$
2	·09	·0961	·1024	2
3	·027	·1207 91	·0327 68	3
4	·0081	·0092 3521	·0104 3576	4
5	·0024 3	·0028 6291 51	33 5544 32	5
6	7 29	8 8750 3681	10 7374 1824	6
7	2 1870	2 7512 6141	3 4359 7384	7
8	..... 6561	..... 8528 9104	1 0995 1163	8
9	..... 1968 3	..... 2643 9622	..... 3518 4372	9
10	..... 0590 49	.. ... 819 6283	..... 1125 8999	10
11	..... 0177 147	..... 254 0848	..... 360 2880	11
12	..... 53 1441	..... 78 7663	..... 115 2921	12
13	..... 15 9432	..... 24 4175	..... 36 8935	13
14	..... 4 7830	..... 7 5694	..... 11 8059	14
15	..... 1 4349	..... 2 3465	..... 3 7779	15
16	..... 4304	..... 7274	..... 1 2089	16
17	..... 1291	..... 2255	..... 3868	17
18	..... 387	..... 699	..... 1238	18
19	..... 116	..... 217	..... 396	19
20	..... 35	..... 67	..... 127	20
21	..... 10	..... 21	..... 40	21
22	..... 3	..... 6	..... 13	22
23		..... 2	..... 4	23
24			..... 1	24

TABLE II. TWELVE DECIMALS.

<i>n</i>	( <i>'33</i> ) <sup><i>n</i></sup>	( <i>'34</i> ) <sup><i>n</i></sup>	( <i>'35</i> ) <sup><i>n</i></sup>	<i>n</i>
2	·1089	·1156	·1225	2
3	·0359 37	·0393 04	·0428 75	3
4	·0118 5927	·0133 6336	·0150 0625	4
5	39 1353 93	45 4334 24	52 5218 75	5
6	12 9146 7969	15 4480 4416	18 3826 5625	6
7	4 2618 4430	5 2523 3501	6 4339 2969	7
8	1 4064 0862	1 7857 9390	2 2518 7539	8
9	..... 4641 1484	..... 6071 6993	..... 7881 5639	9
10	..... 1531 5790	..... 2064 3777	..... 2758 5473	10
11	..... 505 4210	..... 701 8884	..... 965 4916	11
12	..... 166 7889	..... 238 6421	..... 337 9220	12
13	..... 55 0403	..... 81 1383	..... 118 2727	13
14	..... 18 1633	..... 27 5870	..... 41 3954	14
15	..... 5 9939	..... 9 3796	..... 14 4884	15
16	..... 1 9780	..... 3 1890	..... 5 0709	16
17	..... 6527	..... 1 0843	..... 1 7748	17
18	..... 2154	..... 3686	..... 6212	18
19	..... 710	..... 1253	..... 2174	19
20	..... 234	..... 426	..... 761	20
21	..... 77	..... 145	..... 266	21
22	..... 25	..... 49	..... 93	22
23	..... 8	..... 17	..... 33	23
24	..... 3	..... 6	..... 11	24
25	..... 1	..... 2	..... 4	25
26			..... 1	26

TABLE II. TWELVE DECIMALS.

$n$	$(\cdot 36)^n$	$(\cdot 37)^n$	$(\cdot 38)^n$	$n$
2	$\cdot 1296$	$\cdot 1369$	$\cdot 1444$	2
3	$\cdot 0466 \ 56$	$\cdot 0506 \ 53$	$\cdot 0548 \ 72$	3
4	$\cdot 0167 \ 9616$	$\cdot 0187 \ 4161$	$\cdot 0208 \ 5136$	4
5	$60 \ 4661 \ 76$	$69 \ 3439 \ 57$	$79 \ 2351 \ 68$	5
6	$21 \ 7678 \ 2336$	$25 \ 6572 \ 6409$	$30 \ 1093 \ 6384$	6
7	$7 \ 8364 \ 1641$	$9 \ 4931 \ 8771$	$11 \ 4415 \ 5826$	7
8	$2 \ 8211 \ 0991$	$3 \ 5124 \ 7945$	$4 \ 3477 \ 9314$	8
9	$1 \ 0155 \ 9957$	$1 \ 2996 \ 1740$	$1 \ 6521 \ 6101$	9
10	..... $3656 \ 1584$	..... $4808 \ 5844$	..... $6278 \ 2118$	10
11	..... $1316 \ 3170$	..... $1779 \ 1762$	..... $2385 \ 7205$	11
12	..... $473 \ 8381$	..... $658 \ 2952$	..... $906 \ 5738$	12
13	..... $170 \ 5817$	..... $243 \ 5692$	..... $344 \ 4980$	13
14	..... $61 \ 4094$	..... $90 \ 1206$	..... $130 \ 9092$	14
15	..... $22 \ 1074$	..... $33 \ 3446$	..... $49 \ 7455$	15
16	..... $7 \ 9587$	..... $12 \ 3375$	..... $18 \ 9033$	16
17	..... $2 \ 8651$	..... $4 \ 5649$	..... $7 \ 1833$	17
18	..... $1 \ 0314$	..... $1 \ 6890$	..... $2 \ 7296$	18
19	..... ..... $3713$	..... ..... $6249$	..... ..... $1 \ 0373$	19
20	..... ..... $1337$	..... ..... $2312$	..... ..... $3942$	20
21	..... ..... $481$	..... ..... $855$	..... ..... $1498$	21
22	..... ..... $173$	..... ..... $316$	..... ..... $569$	22
23	..... ..... $62$	..... ..... $117$	..... ..... $216$	23
24	..... ..... $22$	..... ..... $43$	..... ..... $82$	24
25	..... ..... $8$	..... ..... $16$	..... ..... $31$	25
26	..... ..... $3$	..... ..... $6$	..... ..... $12$	26
27	..... ..... $1$	..... ..... $2$	..... ..... $4$	27
28			... .. $1,7$	28

TABLE II. TWELVE DECIMALS.

$n$	$(\cdot 39)^n$	$(\cdot 40)^n$	$(\cdot 41)^n$	$n$
2	·1521	·16	·1681	2
3	·0593 19	·064	·0689 21	3
4	·0231 3441	·0256	·0282 5761	4
5	90 2241 99	·0102 4	·0115 8562 01	5
6	35 1874 3761	·0040 96	47 5010 4241	6
7	13 7231 0067	·0016 384	19 4754 2739	7
8	5 3520 0926	6 5536	7 9849 2523	8
9	2 0872 8361	2 6214 4	3 2738 1934	9
10	..... 8140 4061	1 0485 76	1 3422 6593	10
11	..... 3174 7584	..... 4194 304	..... 5503 2903	11
12	..... 1238 1558	..... 1677 7216	..... 2256 3490	12
13	..... 482 8807	..... 671 0886	..... 925 1031	13
14	..... 188 3235	..... 268 4354	..... 379 2923	14
15	..... 73 4462	..... 107 3742	..... 155 5098	15
16	..... 28 6440	..... 42 9497	..... 63 7590	16
17	..... 11 1712	..... 17 1799	..... 26 1412	17
18	..... 4 3567	..... 6 8719	..... 10 7179	18
19	..... 1 6991	..... 2 7487	..... 4 3943	19
20	..... 6626	..... 1 0995	..... 1 8017	20
21	..... 2584	..... 4398	..... 7387	21
22	..... 1008	..... 1759	..... 3029	22
23	..... 393	..... 703	..... 1242	23
24	..... 153	..... 281	..... 509	24
25	..... 60	..... 112	..... 209	25
26	..... 23	..... 45	..... 85	26
27	..... 9	..... 18	..... 35	27
28	..... 3	..... 7	..... 14	28
29	..... 1	..... 3	..... 6	29
30		..... 1	..... 2,4	30



TABLE II. TWELVE DECIMALS.

$n$	$(\cdot 42)^n$	$(\cdot 43)^n$	$(\cdot 44)^n$	$n$
2	·1764	·1849	·1936	2
3	·0740 88	·0795 07	·0851 84	3
4	·0311 1696	·0341 8801	·0374 8096	4
5	·0130 6912 32	·0174 0084 43	·0164 9162 24	5
6	54 8903 1744	63 2136 3049	72 5631 3856	6
7	23 0539 3332	27 1818 6111	31 9277 8097	7
8	9 6826 5200	11 6882 0028	14 0482 2362	8
9	4 0667 1384	5 0259 2612	6 1812 1839	9
10	1 7080 1981	2 1611 4823	2 7197 3609	10
11	..... 7173 6832	..... 9292 9374	1 1966 8388	11
12	..... 3012 9469	..... 3905 9631	..... 5265 4091	12
13	..... 1265 4377	..... 1718 2641	..... 2316 7800	13
14	..... 531 4838	..... 738 8536	..... 1019 3832	14
15	..... 223 2232	..... 317 7070	..... 448 5286	15
16	..... 93 7537	..... 136 6140	..... 197 3526	16
17	..... 30 3766	..... 58 7440	..... 86 8351	17
18	..... 16 5382	..... 25 2599	..... 38 2075	18
19	..... 6 9460	..... 10 8618	..... 16 8113	19
20	..... 2 9173	..... 4 6706	..... 7 3970	20
21	..... 1 2253	..... 2 0083	..... 3 2547	21
22	..... ..... 5146	..... ..... 8636	..... 1 4321	22
23	..... ..... 2161	..... ..... 3713	..... ..... 6301	23
24	..... ..... 908	..... ..... 1597	..... ..... 2772	24
25	..... ..... 381	..... ..... 687	..... ..... 1220	25
26	..... ..... 160	..... ..... 295	..... ..... 537	26
27	..... ..... 67	..... ..... 127	..... ..... 236	27
28	..... ..... 28	..... ..... 54	..... ..... 104	28
29	..... ..... 12	..... ..... 23	..... ..... 46	29
30	..... ..... 5	..... ..... 10	..... ..... 20	30
31	..... ..... 2	..... ..... 4	..... ..... 9	31
32		..... ..... 2	..... ..... 4	32
33			..... ..... $1\frac{2}{3}$	33

TABLE II. TWELVE DECIMALS.

$n$	$(.45)^n$	$(.46)^n$	$(.47)^n$	$n$
2	.2025	.2116	.2209	2
3	.0911 25	.0973 36	.1038 23	3
4	.0410 0625	.0447 7456	.0487 9681	4
5	.0184 5281 2500	.0205 9629 76	.0229 3450 07	5
6	83 0376 5625	94 7429 6896	.0107 7921 5329	6
7	37 3669 4531	43 5817 6572	50 6623 1205	7
8	16 8151 2539	20 0476 1223	23 8112 8666	8
9	7 5668 0642	9 2219 0163	11 1913 0473	9
10	3 4050 6289	4 2420 7475	5 2599 1322	10
11	1 5322 7830	1 9513 5438	2 4721 5921	11
12	..... 6895 2523	..... 8976 2302	1 1619 1483	12
13	..... 3102 8636	..... 4129 0689	..... 5460 9997	13
14	..... 1396 2886	..... 1809 3703	..... 2566 6699	14
15	..... 628 3299	..... 873 7103	..... 1206 2348	15
16	..... 282 7484	..... 401 9067	..... 566 9774	16
17	..... 127 2368	..... 184 8771	..... 266 4794	17
18	..... 57 2566	..... 85 0435	..... 125 2453	18
19	..... 25 7654	..... 39 1200	..... 58 8653	19
20	..... 11 5944	..... 17 9952	..... 27 6667	20
21	..... 5 2175	..... 8 2778	..... 13 0033	21
22	..... 2 3479	..... 3 8078	..... 6 1116	22
23	..... 1 0565	..... 1 7516	..... 2 8724	23
24	..... 4754	..... 8057	..... 1 3500	24
25	..... 2139	..... 3706	..... 6345	25
26	..... 963	..... 1705	..... 2982	26
27	..... 433	..... 784	..... 1402	27
28	..... 193	..... 361	..... 659	28
29	..... 88	..... 166	..... 310	29
30	..... 39	..... 76	..... 145	30
31	..... 18	..... 35	..... 68	31
32	..... 8	..... 16	..... 32	32
33	..... 3,6	..... 7	..... 15	33
34	..... 1,6	..... 3	..... 7	34
35		..... 1,6	..... 3	35
36			..... 1,6	36

TABLE II. TWELVE DECIMALS.

$n$	$(.48)^n$	$(.49)^n$	$(.50)^n$	$n$
2	.2304	.2401	.25	2
3	.1105 92	.1176 49	.125	3
4	.0530 8416	.0576 4801	.0625	4
5	.0254 8039 68	.0282 4752 49	.0312 5	5
6	.0122 3059 0464	.0138 4128 7201	.0156 25	6
7	58 7068 3423	67 8223 0728	.0078 125	7
8	28 1792 8043	33 2329 3057	39 0625	8
9	13 5260 5460	16 2841 3598	19 5312 5	9
10	6 4925 0621	7 9792 2663	9 7656 25	10
11	3 1164 0298	3 9098 2105	4 8828 125	11
12	1 4958 7343	1 9158 1231	2 4414 0625	12
13	..... 7180 1925	..... 9378 4803	1 2207 0312	13
14	..... 3446 4924	..... 4599 8653	..... 6103 5156	14
15	..... 1654 3164	..... 2253 9340	..... 3051 7578	15
16	..... 794 0719	..... 1104 4277	..... 1525 8789	16
17	..... 381 1545	..... 541 1696	..... 0762 9394	17
18	..... 182 9541	..... 263 1731	..... 381 4697	18
19	..... 87 8180	..... 129 9348	..... 190 7348	19
20	..... 42 1526	..... 63 6681	..... 95 3674	20
21	..... 20 2333	..... 31 1975	..... 47 6837	21
22	..... 9 7120	..... 15 2867	..... 23 8418	22
23	..... 4 6618	..... 7 4905	..... 11 9209	23
24	..... 2 2376	..... 3 6703	..... 5 9604	24
25	..... 1 0741	..... 1 7985	..... 2 9802	25
26	..... ..... 5155	..... ..... 8813	..... 1 4901	26
27	..... ..... 2474	..... ..... 4318	..... ..... 7450	27
28	..... ..... 1190	..... ..... 2116	..... ..... 3725	28
29	..... ..... 571	..... ..... 1037	..... ..... 1862	29
30	..... ..... 274	..... ..... 508	..... ..... 931	30
31	..... ..... 132	..... ..... 249	..... ..... 465	31
32	..... ..... 63	..... ..... 122	..... ..... 233	32
33	..... ..... 30	..... ..... 60	..... ..... 116	33
34	..... ..... 14	..... ..... 29	..... ..... 58	34
35	..... ..... 7	..... ..... 14	..... ..... 29	35
36	..... ..... 3	..... ..... 7	..... ..... 14	36
37	..... ..... 1,6	..... ..... 3	..... ..... 7	37
38		..... ..... 1,7	..... ..... 3	38
39			..... ..... 1,8	39

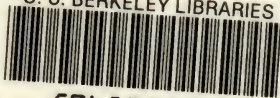








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